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The morphology of $\mathcal{N} = 6$ Chern-Simons theory

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Abstract: We tabulate various properties of the 'language' of $\mathcal{N} = 6$ Chern-Simons Theory, in the sense of Polyakov. Specifically we enumerate and compute character formulas for all syllables of up to four letters, i.e. all irreducible representations of $OSp(6|4)$ built from up to four fundamental fields of the ABJM theory. We also present all tensor product decompositions for up to four singletons and list the (cyclically invariant) four-letter words, which correspond to single-trace operators of length four. As an application of these results we use the two-loop dilatation operator to compute the leading correction to the Hagedorn temperature of the weakly-coupled planar ABJM theory on $\mathbb{R} \times S^2$.

Keywords: Supersymmetric gauge theory, Field Theories in Lower Dimensions, Integrable Equations in Physics, Chern-Simons Theories

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Contents

1 Introduction

It is difficult to overstate the importance of the role that maximally superconformal field theories have played in deepening our understanding of string and field theories, and the relations between them. To date the vast majority of work has focused on four-dimensional gauge theories describing the worldvolume of D3-branes, but much progress has recently been made on three-dimensional theories describing the worldvolume of M2-branes. This progress has been made possible by the discovery [1] (see also [2]) of highly supersymmetric conformal theories in three dimensions, extending earlier attempts [3] with superconformal Chern-Simons theories.

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This paper is concerned with the conformal $\mathcal{N} = 6$ supersymmetric Chern-Simons matter theory constructed by ABJM [4] (see also [5], and [6] for related earlier work). The ABJM theory is a three-dimensional $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge theory with four complex scalars and their fermionic partners in the bi-fundamental representation and gauge fields with Chern-Simons levels $+k$ and $-k$. The theory has a 't Hooft limit in which $N, k \to \infty$ with $\lambda = N/k$ fixed, similar to the story in $\mathcal{N} = 4$ Yang-Mills theory (SYM).

Indeed the strong similarity to SYM has allowed many tools, such as the language of spin chains and methods from integrability (such as [7–10]) which have been so successful in exploring the structure of planar SYM, to be applied to the ABJM theory as well. In particular the anomalous dimensions of local operators in the theory are apparently encoded in an integrable spin chain Hamiltonian (see [11–18] for related recent work), and an exact magnon S-matrix for this spin chain has been proposed in [19] (see also [20]). In several respects however the story of integrability in the ABJM theory is slightly more complicated than that in SYM, one of which (see also [21]) is the fact that anomalous dimensions first show up at two loops. The full two-loop dilatation operator, which has recently been constructed in [22, 23], has both nearest-neighbor and next-to-nearest neighbor interactions, making it necessarily more complicated than the nearest-neighbor one-loop dilatation operator [24] of SYM.

There is a very beautiful and useful analogy [25] between the counting of local gauge-invariant operators in gauge theories such as SYM and linguistics. The elementary fields (and their derivatives) are thought of as 'letters' which are strung together inside single trace operators as 'words', products of which can then be thought of as 'sentences'. Since the ABJM elementary fields transform in either the $(N, \overline{N})$ or the $(\overline{N}, N)$ bi-fundamental representations they must appear in alternating order inside any single-trace operator. If we wish to extend the linguistic analogy to this case we could perhaps say that the ABJM alphabet is divided into consonants and vowels, comprising the respectively the two $OSp(6|4)$ singleton representations (which are conjugate to each other). Every word in the ABJM language has even length and consists of alternating vowels and consonants.

One of the purposes of this paper is to lay some groundwork for detailed spectroscopic analysis of the ABJM theory through two loops using the results of [22, 23]. To this end we first review in section 2 the oscillator construction for $OSp(6|4)$. Then in section 3 we enumerate all syllables of up to four letters (i.e. all irreducible representations of $OSp(6|4)$ built from up to four generations of superoscillators) and also calculate their characters (these may also be found in the exhaustive reference [26]).

In section 4 we present all tensor product decompositions for products of up to four singletons. These tensor product results are useful for analysis of the two-loop dilatation operator since its building block, the "Hamiltonian density" D_2, is an operator which acts simultaneously on three adjacent sites of the spin chain, alternately occupied by the two singletons $\mathcal{V}^1$ and $\overline{\mathcal{V}}^1$. Due to $OSp(6|4)$ symmetry the Hamiltonian density can be written in block-diagonal form with no mixing between irreducible representations of different quantum numbers in the tensor product decomposition of $\mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 \otimes \mathcal{V}^1$. For the four-site tensor products we also calculate the irreducible representations in the tensor product decomposition of $(\mathcal{V}^1 \otimes \overline{\mathcal{V}}^1)^2$ which are symmetric under interchange of the two $\mathcal{V}^1 \otimes$

$\overline{\mathcal{V}}^1$ factors. This representation content corresponds to the physical spectrum of gauge invariant operators of length four in the ABJM theory — i.e. they are the four-letter words in the ABJM language. For completeness we include section 5 in which all of the abovementioned results are tabulated for the $OSp(4|2)$ subsector.

We defer more detailed spectroscopy for later work, only mentioning it here as one motivation for this work. However in section 6 we present a concrete result which follows rather easily from the explicit form of the two-loop dilatation operator and the information presented in this paper: a calculation of the two-loop correction to the Hagedorn temperature $T_{\rm H}$ of the planar ABJM theory on S^2, with the result $\delta \log T_{\rm H} = 2\lambda^2(\sqrt{2}-1)$. Our results might also be useful for studying higher spin symmetry in the free ABJM theory, following for example [27, 28].

2 Oscillator construction for $OSp(6|4)$

In this section we review the oscillator construction for the $OSp(6|4)$ supergroup due to Gunaydin and Hyun [29], together with the particular notations which are convenient for our purposes.[1] We begin with the bosonic $Sp(4,\mathbb{R})$ and SO(6) subgroups before building up to the full $OSp(6|4)$.

2.1 $Sp(4,\mathbb{R}) \simeq SO(3,2)$

The $Sp(4,\mathbb{R}) \simeq SO(3,2)$ generators can be expressed in terms of a set of $f = 2p+\epsilon$ $(\epsilon = 0,1)$ "generations" of bosonic annihilation operators $a_i(r), b_i(r), c_i$, $(i = 1,2, r = 1,2,\ldots,p)$ and their hermitian conjugate creation operators $a^i(r) = a_i^\dagger(r), b^i(r) = b_i^\dagger(r), c^i = c_i^\dagger$, transforming respectively covariantly and contravariantly under U(2). The number of a and b oscillators p can be any integer greater or equal to zero, whereas we only have either zero or one c oscillators, according to the value of ϵ. The oscillators obey the usual commutation commutation relations, the only nonvanishing ones being

$$\big[a_i(r), a^j(s)\big] = \delta_i^j \delta_{rs}\,, \qquad \big[b_i(r), b^j(s)\big] = \delta_i^j \delta_{rs}\,, \qquad \big[c_i, c^j\big] = \delta_i^j\,. \tag{2.1}$$

The $Sp(4,\mathbb{R})$ generators are then given as

$$\begin{aligned} P^{ij} &= \vec{a}^i \cdot \vec{b}^j + \vec{a}^j \cdot \vec{b}^i + \epsilon\, c^i c^j && = \square\!\square\,, \\ K_{ij} &= \vec{a}_i \cdot \vec{b}_j + \vec{a}_j \cdot \vec{b}_i + \epsilon\, c_i c_j && = (P^{ji})^\dagger\,, \\ I^i{}_j &= \vec{a}^i \cdot \vec{a}_j + \vec{b}_j \cdot \vec{b}^i + \frac{\epsilon}{2}\left(c^i c_j + c_j c^i\right) && = (I^j{}_i)^\dagger\,, \end{aligned} \tag{2.2}$$

where we have adopted the vector notation $\vec{a}_i = (a_i(1), a_i(2), \ldots, a_i(p))$ implying $\vec{a}_i \cdot \vec{b}_j = \sum_{r=1}^p a_i(r) b_j(r)$ and so on. The association of P^{ij} with the Young tableau (YT) $\square\!\square$ is included here for convenience and will be explained shortly.

[1] An alternative approach, using the method of Kac, may be found in [30].

In this particular basis the $Sp(4,\mathbb{R})$ algebra is

$$\begin{aligned}
\left[K_{ij}, P^{kl}\right] &= \delta^l_j I^k{}_i + \delta^k_i I^l{}_j + \delta^k_j I^l{}_i + \delta^l_i I^k{}_j \,, \\
\left[I^i{}_j, P^{kl}\right] &= \delta^k_j P^{il} + \delta^l_j P^{ik} \,, \\
\left[I^i{}_j, K_{kl}\right] &= -\delta^i_k K_{jl} - \delta^i_l K_{jk} \,, \\
\left[I^i{}_j, I^k{}_l\right] &= \delta^k_j I^i{}_l - \delta^i_l I^k{}_j \,,
\end{aligned} \tag{2.3}$$

and we can immediately recognize the $I^i{}_j$ as generators of the maximal compact subgroup $\mathrm{U}(2) \subset Sp(4,\mathbb{R})$. In terms of the bosonic number operators which we define as

$$\begin{aligned}
N_{B_i} &= \vec{a}^i \cdot \vec{a}_i + \vec{b}^i \cdot \vec{b}_i + \epsilon\, c^i c_i \quad (\text{no sum on } i)\,, \\
N_B &= N_{B_1} + N_{B_2}\,,
\end{aligned} \tag{2.4}$$

the diagonal entries of the U(2) generators may be rewritten as

$$I^1{}_1 = N_{B_1} + \frac{1}{2} f \,, \qquad I^2{}_2 = N_{B_2} + \frac{1}{2} f \,. \tag{2.5}$$

The form of the commutators (2.3) is essentially the same as that of the three-dimensional conformal group in the spinor basis (see for example [26]) with the identification of

$$\Delta = \frac{1}{2} I^i{}_i = \frac{1}{2}(N_B + f) \tag{2.6}$$

as the dilatation operator[2] and $L^i_j = I^i_j - \delta^i_j\,\Delta$, P^{ij}, and K_{ij} as the generators of rotations, translations, and special conformal transformations respectively.

We are interested in representations of $Sp(4,\mathbb{R})$ for which the spectrum of Δ is bounded from below. Each such representation can be characterized by its lowest-weight state (LWS) $|\Omega\rangle$, a state within the multiplet that

1. is annihilated by all K_{ij}, and
2. transforms irreducibly under the U(2) subgroup.

Of course the "vacuum" $|0\rangle$, defined as usual as the state which is annihilated by all lowering operators $a_i(r), b_i(r), c_i$ is a suitable LWS, but not the only one. The first condition implies that there exist additional lowest-weight states which can be expressed as linear combinations of raising operators acting on $|0\rangle$. Since the raising operators transform in the fundamental two-dimensional representation of U(2), the irreducible representations arising from combinations of M raising operators are related to representations of the permutation group S_M, and the second condition implies that we simply have to symmetrize and antisymmetrize the combinations of raising operators appropriately in order to get a LWS.

[2] The observant reader may worry that Δ is a compact generator, and that in general we construct states that have definite $\mathrm{U}(2) \simeq SO(2) \times \mathrm{SO}(3)$ charges instead of $SO(1,1) \times SO(1,2)$ ones. However it has been proven in [31] that there exists a rotation which maps the eigenstates of one subgroup to those of the other while preserving their eigenvalues, similarly to what happens for the four-dimensional conformal group which had been proven earlier in [32].

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In this manner the U(2) YT description of a LWS arises naturally in conjunction with the actual symmetrization and antisymmetrization of raising operators, and only differs from the proper SU(2) YT in that we do not need to discard two-box columns, as they provide information for the U(1) charge which corresponds to the scaling dimension Δ. If (m_1, m_2) denote the number of boxes in the (first,second) rows of a U(2) YT, then the scaling dimension Δ and the SU(2) spin j of the corresponding LWS are given by

$$(\Delta, j) = \left(\frac{1}{2}(m_1 + m_2 + f), \frac{1}{2}(m_1 - m_2)\right) . \tag{2.7}$$

As an example, it can be shown that for $f = 2$ the only inequivalent $Sp(4, \mathbb{R})$ lowest-weight states are of the form

$$\begin{aligned} a^{i_1} a^{i_2} \cdots a^{i_k} |0\rangle &= \overbrace{\square\square\cdots\square}^{k} & \qquad (\Delta, j) &= \left(1 + \frac{k}{2}, \frac{k}{2}\right) & \qquad k &= 0, 1, 2, \dots , \\ (a^i b^j - a^j b^i)|0\rangle &= \ydiagram{1,1} & \qquad (\Delta, j) &= (2, 0) , \end{aligned} \tag{2.8}$$

where we have also indicated the corresponding Young tableaux and the Cartan charges (Δ, j) of the LWS.

Given the LWS $|\Omega\rangle$ of any representation, a basis R for the entire representation is generated by acting on it with the various P^{ij},

$$R = \{|\Omega\rangle, P^{ij}|\Omega\rangle, P^{lm} P^{ij}|\Omega\rangle, \dots\} . \tag{2.9}$$

There is no restriction on how many times we can act with P^{ij}, hence we produce an an infinite-dimensional representation (a manifestation of the noncompact nature of $Sp(4, \mathbb{R})$). If we are interested in the U(2) content of each of the basis vectors, this can be determined by considering symmetric tensor products of P^{ij}, due to the fact that the P^{ij} commute with each other. Using the identification of P^{ij} with the YT $\ydiagram{2}$ as indicated above in (2.2) we have for example

$$P^{ij} = \ydiagram{2}, \quad (P^{ij})^2_+ = \ydiagram{2,2} + \ydiagram{4}, \quad (P^{ij})^3_+ = \ydiagram{4,2} + \ydiagram{6}, \quad \text{etc,} \tag{2.10}$$

where the subscript $+$ indicates that only the totally symmetric representations in the tensor product decomposition contribute. In this manner one can obtain the full U(2) content of the multiplet by tensoring arbitrarily high symmetric powers of $\ydiagram{2}$ with the YT of the LWS, following the usual decomposition rules.

2.2 SO(6) $\simeq$ SU(4)

The oscillator construction we will be describing here is a particular realization for $n = 3$ of the general method for $SO(2n)$ groups, however since the fundamental fields of the ABJM theory transform in the complex $\mathbf{4}$ and $\bar{\mathbf{4}}$ representations rather than the real $\mathbf{6}$ representation, the language of the locally isomorphic SU(4) will be more suitable for the presentation.

This time the building blocks for the generators will be fermionic oscillators. We again have $f = 2p + \epsilon$, $(\epsilon = 0, 1)$ annihilation operators $\alpha_\mu(r), \beta_\mu(r), \gamma_\mu$, $(\mu = 1, 2, 3,\ r =$

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$1, 2, \ldots, p$) and their hermitian conjugate creation operators $\alpha^\mu(r) = \alpha^\dagger_\mu(r), \beta^\mu(r) = \beta^\dagger_\mu(r)$, $\gamma^\mu = \gamma^\dagger_\mu$ transforming in the conjugate fundamental and fundamental of U(3) respectively. The only nonvanishing anticommutation relations are

$$\{\alpha_\mu(r), \alpha^\nu(s)\} = \delta^\nu_\mu \delta_{rs}\,, \qquad \{\beta_\mu(r), \beta^\nu(s)\} = \delta^\nu_\mu \delta_{rs}\,, \qquad \{\gamma_\mu, \gamma^\nu\} = \delta^\nu_\mu\,. \tag{2.11}$$

With respect to these oscillators the SU(4) generators are given by

$$\begin{aligned} A^{\mu\nu} &= \vec{\alpha}^\mu \cdot \vec{\beta}^\nu - \vec{\alpha}^\nu \cdot \vec{\beta}^\mu + \epsilon\, \gamma^\mu \gamma^\nu &&= \square\,, \\ A_{\mu\nu} &= \vec{\alpha}_\mu \cdot \vec{\beta}_\nu - \vec{\alpha}_\nu \cdot \vec{\beta}_\mu + \epsilon\, \gamma_\mu \gamma_\nu &&= (A^{\nu\mu})^\dagger\,, \\ U^\mu{}_\nu &= \vec{\alpha}^\mu \cdot \vec{\alpha}_\nu - \vec{\beta}_\nu \cdot \vec{\beta}^\mu + \frac{\epsilon}{2}(\gamma^\mu\gamma_\nu - \gamma_\nu\gamma^\mu) &&= (U^\nu{}_\mu)^\dagger\,, \end{aligned} \tag{2.12}$$

from which the corresponding algebra in this particular basis follows:

$$\begin{aligned} [A_{\mu\nu}, A^{\rho\sigma}] &= -\delta^\sigma_\mu U^\rho{}_\nu + \delta^\rho_\mu U^\sigma{}_\nu - \delta^\rho_\nu U^\sigma{}_\mu + \delta^\sigma_\nu U^\rho{}_\mu\,, \\ [U^\mu{}_\nu, A^{\rho\sigma}] &= \delta^\rho_\nu A^{\mu\sigma} + \delta^\sigma_\nu A^{\rho\mu}\,, \\ [U^\mu{}_\nu, A_{\rho\sigma}] &= -\delta^\mu_\rho A_{\nu\sigma} - \delta^\mu_\sigma A_{\rho\nu}\,, \\ [U^\mu{}_\nu, U^\rho{}_\sigma] &= \delta^\rho_\nu U^\mu{}_\sigma - \delta^\mu_\sigma U^\rho{}_\nu\,. \end{aligned} \tag{2.13}$$

It is evident evident from the last line that the $U^\mu{}_\nu$ are generators of a U(3) subgroup.

The relations (2.13) are the so-called split form of the commutation relations because one U(3) is singled out. They can be recast into the standard SU(4) form by defining

$$\begin{aligned} R^\mu{}_\nu &= U^\mu{}_\nu - \frac{1}{2}\delta^\mu_\nu U^\lambda{}_\lambda\,, \quad & R^\mu{}_4 &= +\frac{1}{2}\,\epsilon^{\mu\rho\sigma} A_{\rho\sigma} \quad &\Rightarrow \quad A_{\mu\nu} &= \epsilon_{\mu\nu\rho} R^\rho{}_4\,, \\ R^4{}_4 &= \frac{1}{2} U^\lambda{}_\lambda\,, & R^4{}_\mu &= -\frac{1}{2}\,\epsilon_{\mu\rho\sigma} A^{\rho\sigma} &\Rightarrow \quad A^{\mu\nu} &= \epsilon^{\mu\nu\rho} R^4{}_\rho\,, \end{aligned} \tag{2.14}$$

which as a consequence of (2.13) obey

$$[R^\alpha{}_\beta, R^\gamma{}_\delta] = \delta^\gamma_\beta R^\alpha{}_\delta - \delta^\alpha_\delta R^\gamma{}_\beta\,, \qquad \text{where here } \alpha, \beta, \gamma, \delta = 1, \ldots, 4\,, \tag{2.15}$$

with $R^\alpha{}_\alpha = 0$, $(R^\beta{}_\alpha)^\dagger = R^\alpha{}_\beta$ as required for SU(4).

For future use we mention here the forms of the diagonal U and R generators in terms of the fermionic number operators

$$\begin{aligned} N_{F_\mu} &= \vec{\alpha}^\mu \cdot \vec{\alpha}_\mu + \vec{\beta}^\mu \cdot \vec{\beta}_\mu + \epsilon\, \gamma^\mu \gamma_\mu \quad (\text{no sum on } \mu)\,, \\ N_F &= N_{F_1} + N_{F_2} + N_{F_3} \end{aligned} \tag{2.16}$$

which are

$$\begin{aligned} U^\mu{}_\mu &= N_{F_\mu} - \frac{1}{2} f \qquad && (\text{no sum on } \mu)\,, \\ R^\mu{}_\mu &= N_{F_\mu} - N_F + \frac{1}{4} f && (\text{no sum on } \mu)\,, \\ R^4{}_4 &= N_F - \frac{3}{4} f\,. \end{aligned} \tag{2.17}$$

As the oscillator method for the construction of representations is very general and applies to a variety of groups and supergroups, see for example [29, 33], the essential features of the SU(4) case are similar to what we saw for the $Sp(4,\mathbb{R})$ representations in the previous section. In a nutshell, after we define the "vacuum" $|0\rangle$ as the state annihilated by all lowering operators $\alpha_\mu(r), \beta_\mu(r), \gamma_\mu$, we look for lowest-weight states $|\Omega\rangle$ that are annihilated by $A_{\mu\nu}$ and transform irreducibly under U(3). These are expressed in terms of properly symmetrized and antisymmetrized creation operators, described naturally in terms of U(3) Young tableaux, whose only difference from the proper SU(3) Young tableaux is that we no longer discard three-box columns.

Then by acting repeatedly on a LWS with various $A^{\mu\nu}$ we build a basis for an irreducible representation of SU(4), and the U(3) content of each of the representation may be found by tensoring the symmetric powers of $A^{\mu\nu}$,

$$A^{\mu\nu} = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\,, \qquad (A^{\mu\nu})^2_+ = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}\,, \qquad (A^{\mu\nu})^3_+ = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline\end{array}\,, \tag{2.18}$$

and so on, with the YT of the LWS.

The only major difference compared to the $Sp(4,\mathbb{R})$ case is that since we have $3f$ fermionic oscillators, $(A^{\mu\nu})^k = 0$ for $k > \frac{3}{2}f$ and hence the representations will now be finite-dimensional, reflecting the compactness of SU(4). Therefore each representation has a highest-weight state (HWS) which is annihilated by all $A^{\mu\nu}$, transforms irreducibly under U(3), and is related to the LWS by unitarity. We will make use of this relation, as the labels of a representation are related to the weights of its HWS. In particular, we will use SU(4) Dynkin labels to characterize representations,[3] and as we'll see later on these are related to the labels (l_1, l_2, l_3) denoting the number of boxes in the first, second and third rows of the U(3) LWS YT by

$$[d_1, d_2, d_3] = [f - l_1 - l_2, l_2 - l_3, l_1 - l_2]\,. \tag{2.19}$$

We should mention that in the literature another labeling convention is also widely used (for example in [26, 34]) which is based on eigenvalues under rotations in three orthogonal planes in $\mathbb{R}^6$ (hence more suited to the SO(6) description of the algebra), known as SO(6) Gelfand-Zetlin labels (r_1, r_2, r_3) . A basic property they obey is that $r_1 \geq r_2 \geq |r_3|$, and in our conventions they are related to the SU(4) Dynkin labels by

$$(r_1, r_2, r_3) = \left(d_2 + \frac{1}{2}(d_3 + d_1), \frac{1}{2}(d_3 + d_1), \frac{1}{2}(d_3 - d_1)\right)\,. \tag{2.20}$$

We illustrate the basic steps of the procedure described above with an example. For $f = 1$ the only possible lowest-weight states and Young tableaux are[4]

$$\begin{aligned} |0\rangle &= 1 \qquad & [d_1, d_2, d_3] &= [1, 0, 0] \qquad & (r_1, r_2, r_3) &= \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)\,, \\ \gamma^\mu|0\rangle &= \square \qquad & [d_1, d_2, d_3] &= [0, 0, 1] \qquad & (r_1, r_2, r_3) &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\,, \end{aligned} \tag{2.21}$$

where we have also indicated both the Dynkin and Gelfand-Zetlin labels of the corresponding representations.

[3] SU(4) and SO(6) have the same Dynkin diagram with just the ordering of the first two roots switched, which translates into the relation $[d_1, d_2, d_3]^{\mathrm{SU}(4)} = [d_2, d_1, d_3]^{\mathrm{SO}(6)}$ for their Dynkin labels.

[4] We use "1" to denote the singlet of the U(3) subgroup, not the singlet of the full group.

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2.3 $OSp(6|4)$ and super-Young tableaux

For the full $OSp(6|4)$ superalgebra one needs to combine the $Sp(4,\mathbb{R})$ and SO(6) oscillators described in the previous sections into U(2|3) contravariant and covariant superoscillators as follows:

$$
\begin{aligned}
\xi_A(r) &= \begin{pmatrix} a_i(r) \\ \alpha_\mu(r) \end{pmatrix}, & \xi^A(r) &= \xi_A(r)^\dagger = \begin{pmatrix} a^i(r) \\ \alpha^\mu(r) \end{pmatrix} = \boxslash, \\
\eta_A(r) &= \begin{pmatrix} b_i(r) \\ \beta_\mu(r) \end{pmatrix}, & \eta^A(r) &= \eta_A(r)^\dagger = \begin{pmatrix} b^i(r) \\ \beta^\mu(r) \end{pmatrix} = \boxslash, \\
\zeta_A &= \begin{pmatrix} c_i \\ \gamma_\mu \end{pmatrix}, & \zeta^A &= \zeta_A{}^\dagger = \begin{pmatrix} c^i \\ \gamma^\mu \end{pmatrix} = \boxslash,
\end{aligned}
\tag{2.22}
$$

where the super-index A takes the values $1,2|1,2,3$ and $r = 1,\ldots,p$. The nonvanishing super-commutation relations are

$$
\big[\xi_A(r), \xi^B(s)\big\} = \delta_A^B \delta_{rs}\,, \qquad \big[\eta_A(r), \eta^B(s)\big\} = \delta_A^B \delta_{rs}\,, \qquad \big[\zeta_A, \zeta^B\big\} = \delta_A^B\,, \tag{2.23}
$$

where the super-commutators are defined as

$$
\big[\xi_A(r), \xi^B(s)\big\} = \xi_A(r)\xi^B(s) - (-1)^{(\deg A)(\deg B)} \xi^B(s)\xi_A(r)\,, \tag{2.24}
$$

etc., with $\deg A = 0$ ($\deg A = 1$) if A is a bosonic (fermionic) index.

The $OSp(6|4)$ generators can then be realized as bilinears of these superoscillators:

$$
\begin{aligned}
S^{AB} &= \vec{\xi}^A \cdot \vec{\eta}^B + \vec{\eta}^A \cdot \vec{\xi}^B + \epsilon\, \zeta^A \zeta^B && = \boxslash\boxslash\,, \\
S_{AB} &= \vec{\xi}_A \cdot \vec{\eta}_B + \vec{\eta}_A \cdot \vec{\xi}_B + \epsilon\, \zeta_A \zeta_B && = (S^{BA})^\dagger\,, \\
M^A{}_B &= \vec{\xi}^A \cdot \vec{\xi}_B + (-1)^{(\deg A)(\deg B)} \vec{\eta}_B \cdot \vec{\eta}^A && \\
&\quad + \frac{\epsilon}{2}\Big(\zeta^A \zeta_B + (-1)^{(\deg A)(\deg B)} \zeta_B \zeta^A\Big) && = (M^B{}_A)^\dagger\,.
\end{aligned}
\tag{2.25}
$$

Of course by restricting to purely bosonic or fermionic indices we recover the generators of the bosonic subgroups

$$
\begin{aligned}
&(M^i{}_j = I^i{}_j, S_{ij} = K_{ij}, S^{ij} = P^{ij}) && \leftrightarrow \quad Sp(4,\mathbb{R})\,, \\
&(M^\mu{}_\nu = U^\mu{}_\nu, S_{\mu\nu} = A_{\mu\nu}, S^{\mu\nu} = A^{\mu\nu}) && \leftrightarrow \quad \mathrm{SO}(6)\,.
\end{aligned}
\tag{2.26}
$$

Whenever the indices take specific integer values we will use the notation of the previous sections for these bosonic generators.

The odd generators have one bosonic and one fermionic index, and we will use the relation $(M^A{}_B)^\dagger = M^B{}_A$ to always display the bosonic index to the left and the fermionic index to the right. The anticommutators among the odd generators are explicitly given by

$$
\begin{aligned}
\{S_{i\mu}, S^{j\nu}\} &= \delta_\mu^\nu I^j{}_i - \delta_i^j U^\nu{}_\mu\,, & \{S_{i\mu}, M^j{}_\nu\} &= -\delta_i^j A_{\mu\nu}\,, \\
\{M^i{}_\mu, M_j{}^\nu\} &= \delta_\mu^\nu I^i{}_j + \delta_j^i U^\nu{}_\mu\,, & \{S_{i\mu}, M_j{}^\nu\} &= \delta_\mu^\nu K_{ij}\,,
\end{aligned}
\tag{2.27}
$$

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together with others obtained by hermitian conjugation. Finally, the commutators between even and odd oscillators are

$$\begin{aligned}
\left[I^i{}_j, M^k{}_\mu\right] &= \delta^k_j M^i{}_\mu\,, & \left[U^\mu{}_\nu, M^k{}_\lambda\right] &= -\delta^\mu_\lambda M^k{}_\nu\,,\\
\left[I^i{}_j, M_k{}^\mu\right] &= -\delta^i_k M_j{}^\mu\,, & \left[U^\mu{}_\nu, M_k{}^\lambda\right] &= \delta^\lambda_\nu M_k{}^\mu\,,\\
\left[I^i{}_j, S_{k\mu}\right] &= -\delta^i_k S_{j\mu}\,, & \left[U^\mu{}_\nu, S_{k\lambda}\right] &= -\delta^\mu_\lambda S_{k\nu}\,,\\
\left[I^i{}_j, S^{k\mu}\right] &= \delta^k_j S^{i\mu}\,, & \left[U^\mu{}_\nu, S^{k\lambda}\right] &= \delta^\lambda_\nu S^{k\mu}\,,\\
\left[K_{ij}, M^k{}_\mu\right] &= \delta^k_i S_{j\mu} + \delta^k_j S_{i\mu}\,, & \left[A_{\mu\nu}, M^\lambda{}_k\right] &= -\delta^\lambda_\mu S_{\nu k} + \delta^\lambda_\nu S_{\mu k}\,,\\
\left[K_{ij}, S^{k\mu}\right] &= \delta^k_i M_j{}^\mu + \delta^k_j M_i{}^\mu\,, & \left[A_{\mu\nu}, S^{k\lambda}\right] &= -\delta^\lambda_\mu M^k{}_\nu + \delta^\lambda_\nu M^k{}_\mu\,,
\end{aligned} \tag{2.28}$$

where we have again omitted commutators which can be obtained from these by hermitian conjugation. From the above relations we can easily determine the dilatation charges of the odd generators,

$$[\Delta, J] = C(J)\, J\,, \tag{2.29}$$

where $C(J) = \frac{1}{2}$ for $J = M^k{}_\mu, S^{k\mu}$ and $C(J) = -\frac{1}{2}$ for $M_k{}^\mu, S_{k\mu}$, indicating that these correspond respectively to the 6 supersymmetry and 6 superconformal generators.

The procedure for the construction of $OSp(6|4)$ representations is a refinement of what we saw for the bosonic sectors. We define the vacuum $|0\rangle$ of the Fock space of states to be the state which is annihilated by the covariant oscillators $\xi_A(r), \eta_A(r), \zeta_A$, and then consider lowest-weight states which are annihilated by S_{AB} and transform irreducibly under $U(2|3)$. A non-exhaustive list of states that satisfy the first condition is given by [35]

$$[\zeta^A]^{k_0}[\xi^B(1)]^{k_1}\cdots[\xi^C(r)]^{k_r}[\eta^D(r+1)]^{k_{r+1}}\cdots[\eta^E(p)]^{k_p}|0\rangle \tag{2.30}$$

and

$$\left(\xi^A(r)\eta^B(r) - \eta^A(r)\xi^B(r)\right)|0\rangle\,, \tag{2.31}$$

where $r = 1, 2, \ldots, p$, $k_0 = 0, 1$ and the other k_i are nonnegative integers. As we'll see in the next section, there exist a few more possibilities for states that can be annihilated by S_{AB}, but these two types capture the great majority of possible lowest-weight states.

The new elements when we look at supergroups come from the condition of irreducibility, and from the fact that we are promoting the oscillators to superoscillators. In particular, in order for linear combinations of superoscillators to transform irreducibly under $U(2|3)$, it is necessary to demand that if the superindices are symmetrized for bosonic values, they have to be antisymmetrized for fermionic values, and vice versa. This can be seen heuristically by convincing oneself that $\xi^A\xi^B$ transforms irreducibly, but when both indices are bosonic (fermionic) it is automatically (anti-) symmetrized.

Thus we define graded symmetrization or "supersymmetrization"

$$\xi^{(A}\eta^{B)} \equiv \xi^A\eta^B + \eta^A\xi^B = \xi^A\eta^B + (-1)^{(\deg A)(\deg B)}\xi^B\eta^A\,, \tag{2.32}$$

and graded antisymmetrization or "superantisymmetrization"

$$\xi^{[A}\eta^{B]} \equiv \xi^A\eta^B - \eta^A\xi^B = \xi^A\eta^B - (-1)^{(\deg A)(\deg B)}\xi^B\eta^A\,, \tag{2.33}$$

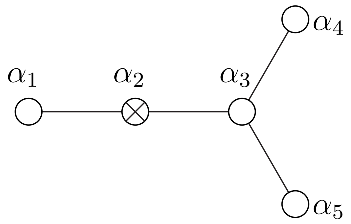

Figure 1. The $OSp(6|4)$ Dynkin diagram.

with the obvious extensions of these definitions to products of more than two superoscillators. Products of superoscillators with all their indices either supersymmetrized and/or superantisymmetrized will form irreducible representations of $U(2|3)$ [36], and these can categorized in terms of super-Young tableaux (SYT) in the same manner that symmetrizations and/or antisymmetrizations of $\mathrm{U}(N)$ oscillators are categorized by ordinary Young tableaux.

There exists a rich literature on super-Young tableaux [36–41] and their applications (see for example [29, 32, 33, 35, 42]), so here we'll just mention a few intuitive examples. To any contravariant superoscillator, namely the ones of the second row of (2.22), there corresponds single (slashed) SYT $\boxslash$. For the two graded symmetrized (2.32) and antisymmetrized (2.33) oscillators we have the SYT $\boxslash\boxslash$ and $\begin{smallmatrix}\boxslash\\\boxslash\end{smallmatrix}$ respectively. Consequently k oscillators of the same kind $\xi^{A_1}\cdots\xi^{A_k}$ will be described by a SYT with a single row of k boxes, $\boxslash\cdots\boxslash$, and for mixed products we first supersymmetrize all superoscillators corresponding to each row, and then superantisymmetrize with respect to the columns. All of these properties are analogous to ordinary Young tableaux, however a major difference is unlike ordinary Young tableaux a SYT can have any number or rows as a consequence of the fact that graded antisymmetrization corresponds to symmetrization of fermionic indices, and hence can be carried out indefinitely.

Returning to our discussion of $OSp(6|4)$ representations, the lesson is that we can obtain all states of a multiplet by tensoring the SYT of its LWS with arbitrary supersymmetrized powers of $S^{AB} = \boxslash\boxslash$. The SYT for the latter just follows by comparing its oscillator form in (2.25) with (2.32). Since each $U(2|3)$ representation decomposes into a set of "component" representations of the bosonic subgroup $\mathrm{U}(2)\times\mathrm{U}(3)$, one way to proceed with the tensoring is to first perform the decomposition and then tensor the ordinary Young tableaux according to the usual rules.

We defer the details of the decomposition of $U(2|3)$ states to the appendix and we provide information about the tensoring procedure in section 3.2. Furthermore, in section 3.1 we will give the explicit relation between the SYT labeling of supermultiplets and other widely used conventions.

2.4 Serre-Chevalley basis

In this section we study the structure of the $OSp(6|4)$ algebra and determine the Cartan charges of each state in the oscillator construction. As a useful application we also provide the representation labels for each class of solutions of the ABJM theory's two-loop Bethe ansatz [11], a result also presented in [23] in a slightly different manner.

The $OSp(6|4)$ Dynkin diagram in the distinguished basis is shown in figure 1, from which the Cartan matrix[5]

$$\mathcal{K} = \left(\begin{array}{c|c|ccc} 2 & -1 & & & \\ \hline -1 & & +1 & & \\ \hline & -1 & +2 & -1 & -1 \\ & & -1 & +2 & \\ & & -1 & & +2 \end{array}\right) \tag{2.34}$$

follows. It is evident both from the Dynkin diagram and the Cartan matrix that the roots $\alpha_3, \alpha_4, \alpha_5$ belong to the $\mathrm{SO}(6) \subset OSp(6|4)$ subgroup while α_1 corresponds to the $\mathrm{SU}(2) \subset Sp(4,\mathbb{R}) \subset OSp(6|4)$ subgroup.

We would like to express the $OSp(6|4)$ superalgebra in a Serre-Chevalley basis (see for example [43]), which is defined by the relations

$$\begin{aligned} [H_i, H_j] &= 0\,, \\ \Big[H_i, E_j^{\pm}\Big] &= \pm\mathcal{K}_{ij} E_j^{\pm}\,, \\ \Big[E_i^{+}, E_j^{-}\Big\} &= H_i \delta_{ij}\,, \\ \{E_i^{\pm}, E_j^{\pm}\} &= 0 \qquad\qquad \text{if } \mathcal{K}_{ii} = 0\,, \\ (ad_{E_i^{\pm}})^{1-\tilde{\mathcal{K}}_{ij}} E_j^{\pm} &= 0\,, \end{aligned} \tag{2.35}$$

where $\mathcal{K}_{ij}$ are the matrix elements of the Cartan matrix $\mathcal{K}$, $\tilde{\mathcal{K}} = (\tilde{\mathcal{K}}_{ij})$ is deduced from $\mathcal{K}$ by replacing all its positive off-diagonal entries by -1, and the last line means $(1 - \tilde{\mathcal{K}}_{ij})$ times the adjoint action of $E_i^{\pm}$ on $E_j^{\pm}$, which in turn is defined as

$$(ad_x)y = [x, y\}\ , \qquad (ad_x)^2 y = [x, [x, y\}\}\ , \qquad \text{etc.} \tag{2.36}$$

The type of the Dynkin diagram also requires supplementary conditions to be imposed around the fermionic root α_2 [43],

$$(ad_{E_2^{\pm}})(ad_{E_3^{\pm}})(ad_{E_2^{\pm}})E_1^{\pm} = (ad_{E_2^{\pm}})(ad_{E_1^{\pm}})(ad_{E_2^{\pm}})E_3^{\pm} = 0\,. \tag{2.37}$$

Starting from (2.3), (2.13), (2.27) and (2.28), the second and third relations of (2.35) together with the Cartan matrix information (2.34) are essentially sufficient for determining H_i and $E_i^{\pm}$, and then the remaining relations of (2.35) and (2.37) can be readily verified.

[5]We use the standard convention where the nonzero diagonal elements are always equal to 2. In many cases where the Cartan matrix appears in relation to the Bethe ansatz, an alternative convention is also used where the rows corresponding to the fermionic and conformal Cartan generators have their signs flipped, see for example [8, 11]. This is permissible due to the invariance of the Bethe ansatz under this inversion.

In this way we find that the Cartan charges are given by

$$\begin{aligned}
H_1 &= I^2{}_2 - I^1{}_1 = N_{B_2} - N_{B_1}\,, \\
H_2 &= I^1{}_1 + U^3{}_3 = N_{B_1} + N_{F_3}\,, \\
H_3 &= U^3{}_3 - U^2{}_2 = N_{F_3} - N_{F_2}, \\
H_4 &= U^2{}_2 - U^1{}_1 = N_{F_2} - N_{F_1}\,, \\
H_5 &= U^2{}_2 + U^1{}_1 = N_{F_1} + N_{F_2} - f\,,
\end{aligned} \tag{2.38}$$

whereas the corresponding raising/lowering operators are

$$\begin{aligned}
E_1^+ &= I^2{}_1\,, & E_1^- &= I^1{}_2\,, \\
E_2^+ &= M^1{}_3\,, & E_2^- &= M_1{}^3\,, \\
E_3^+ &= U^3{}_2\,, & E_3^- &= U^2{}_3\,, \\
E_4^+ &= U^2{}_1\,, & E_5^- &= U^1{}_2\,, \\
E_5^+ &= A^{21}\,, & E_4^- &= A_{12}\,,
\end{aligned} \tag{2.39}$$

and we see that by construction $(E_i^+)^\dagger = E_i^-$. As an independent check on this result we considered the Chevalley bases of $Sp(4,\mathbb{R})$ and SU(4) separately and obtained the fermionic Cartan charge from them according to [44], finding agreement with the above straightforward calculation.

Once we have the Serre-Chevalley basis it is easy to establish the relation between the weight of any given state and its excitation numbers (i.e., the numbers of each type of raising operators needed to reach it from a LWS). In particular, given that the LWS annihilated by all E_i^- is

$$|\Omega\rangle \equiv \big(|0\rangle \otimes \gamma^1|0\rangle\big)^L\,, \tag{2.40}$$

then an arbitrary state

$$(E_1^+)^{K_w}(E_2^+)^{K_s}(E_3^+)^{K_r}(E_4^+)^{K_v}(E_5^+)^{K_u}|\Omega\rangle \tag{2.41}$$

(the choice of notation for the K's here anticipates the connection with that of [11]) will have number operator eigenvalues

$$\begin{aligned}
N_{B_1} &= K_s - K_w\,, & H_1 &= 2K_w - K_s\,, \\
N_{B_2} &= K_w\,, & H_2 &= K_r - K_w\,, \\
N_{F_1} &= L + K_u - K_v\,, & H_3 &= 2K_r - K_u - K_v - K_s\,, \\
N_{F_2} &= K_u + K_v - K_r\,, & H_4 &= 2K_v - K_r - L\,, \\
N_{F_3} &= K_r - K_s\,, & H_5 &= 2K_u - K_r - L\,.
\end{aligned} \tag{2.42}$$

The right-hand sides of the above relations give us the weights of a state with excitation numbers $\{K_w, K_s, K_r, K_v, K_u\}$. In particular H_1 is twice the $\mathrm{SU}(2) \subset Sp(4,\mathbb{R})$ spin and $[H_4, H_3, H_5]$ are the SU(4) weights in the Dynkin basis. Furthermore if we assume their values refer to the the LWS of a certain $OSp(6|4)$ supermultiplet, then by unitarity the labels of the corresponding HWS will simply be[6] $j = -H_1/2 = (N_{B_1} - N_{B_2})/2$ and $[d_1, d_2, d_3] = [-H_5, -H_3, -H_4] = [f - N_{F_1} - N_{F_2}, N_{F_2} - N_{F_3}, N_{F_1} - N_{F_2}]$. Together with the scaling dimension (2.6) and the number of generations f, these comprise the Dynkin labels of the supermultiplet in question, and we have shown that they are related to the excitation numbers of the LWS as

$$\begin{aligned} &\Delta = L + \frac{1}{2}K_s\,, && j = \frac{1}{2}K_s - K_w\,, && f = 2L\,, \\ &d_1 = L + K_r - 2K_u\,, && d_2 = K_v + K_u - 2K_r + K_s\,, && d_3 = L + K_r - 2K_v\,, \end{aligned} \tag{2.43}$$

in exact agreement with [23]. Since the excitation numbers K are the same quantities that appear in the theory's Bethe ansatz [11] (denoting the number of Bethe roots of each type which appear in a solution of the Bethe equations) this formula is useful in identifying which symmetry multiplet any particular solution belongs to.

3 Representations and their partition functions

3.1 Notational conventions

We have seen that each $OSp(6|4)$ supermultiplet can be characterized by the number f of generations of superoscillators used in realizing the generators, see (2.25), and the SYT of its LWS. We will use $\mathcal{V}^f$ to denote the representation with f generations whose LWS is $|0\rangle$. For a more general $OSp(6|4)$ supermultiplet whose LWS transforms in a non-trivial representation of $U(2|3)$ we indicate the SYT of that $U(2|3)$ representation as a subscript on $\mathcal{V}^f$. Thus the representation with $f = 2$ and LWS $\xi^A \cdots \xi^B|0\rangle = \boxslash\boxslash\cdots\boxslash$ will be denoted $\mathcal{V}^2_{\boxslash\boxslash\cdots\boxslash}$, and so on.

The information provided graphically by the SYT can equally well be encoded in labels $(k_1, k_2, \ldots)$ which indicate the number of boxes in the (first, second, $\ldots$) row of the tableau. Note that since the number of rows is not fixed, as we saw in section 2.3, the number of labels will also vary for a given f. For example we can have $\mathcal{V}^f_{\boxslash\boxslash} = \mathcal{V}^f_2$, $\mathcal{V}^f_{\substack{\boxslash\boxslash\boxslash \\ \boxslash\boxslash}} = \mathcal{V}^f_{3,2}$, etc.

More conventionally, a multiplet may alternatively be described by the eigenvalues of some set of states within the multiplet under the action of the generators of the Cartan subalgebra. The simplest choice is to use the labels of the bosonic subgroup $Sp(4,\mathbb{R})\times\mathrm{SO}(6)$ we saw earlier, namely the scaling dimension Δ, the spin in a particular axis in $\mathrm{SO}(3) \subset SO(3,2)$, denoted by j, and the SU(4) Dynkin labels $[d_1, d_2, d_3]$. As far as choosing a particular state whose charges will be used to label the entire multiplet, we pick a primary state, defined as a state annihilated by the superconformal generators, or alternatively as a state with the lowest scaling dimension in the multiplet. In order to have positive labels we also demand that the state is of highest-weight type with respect to both $\mathrm{SO}(3) \subset SO(3,2)$ and SU(4). However as we saw in section 2.4 all the information can be obtained by the appropriate LWS by unitarity.

[6]These expressions of the representation labels in terms of number operators also make contact with and explain (2.7) and (2.19). The value of j is determined by the state whose U(2) YT has only oscillators with index 1 on the first row and only oscillators with index 2 on the second row. An analogous statement holds for the SU(4) labels.

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So the translation between the SYT labeling and the Cartan labeling of $OSp(6|4)$ supermultiplets simply consists of finding the $Sp(4,\mathbb{R})\times \mathrm{SO}(6)$ submultiplet with the smallest number of $\mathrm{U}(2) \subset Sp(4,\mathbb{R})$ boxes in the respective decomposition (see the appendix) and calculating its Cartan labels with the help of (2.7) and (2.19). We mention them here combined for convenience:

$$[\Delta, j; d_1, d_2, d_3] = \left[\frac{1}{2}(m_1 + m_2 + f), \frac{1}{2}(m_1 - m_2); f - l_1 - l_2, l_2 - l_3, l_1 - l_2\right] . \tag{3.1}$$

We will use $\mathcal{V}^f_{(\Delta,j)[d_1,d_2,d_3]}$ to denote the representation with the given labels. We will also use a bar to denote the 'conjugate' of a representation, which is obtained by exchanging two of the SU(4) labels: $\overline{\mathcal{V}}^f_{(\Delta,j)[d_1,d_2,d_3]} = \mathcal{V}^f_{(\Delta,j)[d_3,d_2,d_1]}$.

The procedure of finding the $Sp(4,\mathbb{R}) \times \mathrm{SO}(6)$ submultiplet with the lowest scaling dimension that we described above can in fact be performed for an arbitrary SYT, thus yielding a general formula to relate the Cartan labels of supermultiplet and its SYT labels $(k_1, k_2, \ldots, k_n)$. The result is that, for $k_n \leq \ldots \leq k_3 \leq 3$,

$$V^f_{k_1,k_2,\ldots,k_n} = \mathcal{V}^f_{(\Delta,j)[d_1,d_2,d_3]} \tag{3.2}$$

with

$$\begin{aligned} \Delta &= \frac{1}{2}\left(\max(k_1 - 3, 0) + \max(k_2 - 3, 0) + f\right) , \\ j &= \frac{1}{2}(\max(k_1 - 3, 0) - \max(k_2 - 3, 0)) , \\ d_1 &= f - \sum_{i=1}^{n} \min(k_i, 2) , \\ d_2 &= \sum_{i=1}^{n} \delta_{k_i,2} , \\ d_3 &= \sum_{i=1}^{n} \delta_{k_i,1} . \end{aligned} \tag{3.3}$$

For more information about the relation between SYT and Dynkin labels see [39].

Finally we will also make use of characters for $OSp(6|4)$ representations. We define the characters to weigh states according to the Cartan charges (3.1), specifically

$$\chi_{\mathcal{V}}(x, y, u, r, v) = \mathrm{Tr}_{\mathcal{V}}[x^\Delta y^j u^{d_1} r^{d_2} v^{d_3}] , \tag{3.4}$$

given by the trace over all states of a particular $OSp(6|4)$ representation $\mathcal{V}$. Explicit formulas for all $OSp(6|4)$ characters may be found in [26], so in order to save space we will for the most part only display explicit formulas for the 'partition functions'

$$V(x) = \mathrm{Tr}_{\mathcal{V}}[x^\Delta] = \chi_{\mathcal{V}}(x, 1, 1, 1, 1) \tag{3.5}$$

which count the number of states within the multiplet $\mathcal{V}$ for each value of the classical scaling dimension Δ.

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3.2 Calculating characters: an example

In this section we demonstrate by a a particular example the steps for calculating the character of a general $OSp(6|4)$ representation using the oscillator construction. At the end of the day we will focus on the partition functions defined in (3.5), and show how the general formulas reduce to those. The reader interested in the final answer may jump to the next section, where the most relevant partition functions are summarized.

The basic computational strategy exploits the fact that an $OSp(6|4)$ supermultiplet decomposes into multiplets of the bosonic subgroup in order to express its character in terms of a sum of $Sp(4,\mathbb{R}) \times \mathrm{SO}(6)$ characters. Due to the noncompactness of $Sp(4,\mathbb{R})$ we also reduce the calculation of its character to a sum of the characters of its maximal compact subgroup U(2). For simplicity we initially sum over the U(2) multiplets that don't contain spacetime derivatives (generated by P^{ij}), and account for the missing states only at the very end.

Finding the bosonic lowest-weight states is a two-step process. First, we decompose the LWS of the supermultiplet according to the rules tabulated in the appendix. The occurring states are annihilated by all of the S_{AB}, so they certainly are lowest-weight states of the $Sp(4,\mathbb{R}) \times \mathrm{SU}(4)$ bosonic subgroup as well, but there are additional $Sp(4,\mathbb{R}) \times \mathrm{SU}(4)$ lowest weight states which are annihilated by all of the K_{ij} and $A_{\mu\nu}$ but not by all of the S_{AB}.

We find the remaining $Sp(4,\mathbb{R}) \times \mathrm{SU}(4)$ lowest-weight states by acting on the supermultiplet LWS with the odd generators $S^{i\mu}$. Once we have obtained all the lowest-weight states of the bosonic subgroup with this method we can then find the labels $[\Delta, s; d_1, d_2, d_3, f]$ of their corresponding submultiplets with the help of (3.3).

As far as the action of the odd raising operators on the supermultiplet LWS is concerned, we use the fact that the $S^{i\mu}$ transform in the (fundamental,fundamental) of the $\mathrm{U}(2) \times \mathrm{U}(3)$ subgroup of $Sp(4,\mathbb{R}) \times \mathrm{SO}(6)$, with corresponding YT $(\yng(1),\yng(1))$. Also, since the $S^{i\mu}$ generators anticommute with each other we can have nonvanishing products of no more than six of them, and those products will transform in antisymmetrized powers of $(\yng(1),\yng(1))$. We can thus write

$$\begin{aligned}
S^{i\mu} &= (\yng(1),\yng(1)) \,, \\
\left(S^{i\mu}\right)^2_- &= (\yng(2),\yng(1,1)) + (\yng(1,1),\yng(2)) \,, \\
\left(S^{i\mu}\right)^3_- &= \left(\yng(3),\yng(1,1,1)\right) + (\yng(2,1),\yng(2,1)) \,, \\
\left(S^{i\mu}\right)^4_- &= \left(\yng(3,1),\yng(2,1,1)\right) + (\yng(2,2),\yng(2,2)) \,, \\
\left(S^{i\mu}\right)^5_- &= \left(\yng(3,2),\yng(2,2,1)\right) \,, \\
\left(S^{i\mu}\right)^6_- &= \left(\yng(3,3),\yng(2,2,2)\right) \,,
\end{aligned} \tag{3.6}$$

and the $Sp(4,\mathbb{R}) \times \mathrm{SO}(6)$ multiplets arising from the action of the $S^{i\mu}$ generators on the $OSp(6|4)$ supermultiplet may be deduced with the help of (3.6), the decompositions of the appendix and the usual rules for tensor products of $\mathrm{U}(2) \times \mathrm{U}(3)$ Young tableaux.

Finally, a subtlety that sometimes arises is that the U(3) Young tableaux resulting from the tensor products may lead to SU(4) Dynkin labels where one of the entries is negative, due to the formula $d_1 = f - l_1 - l_2$. These representations may either vanish, or be mapped to representations with positive labels, after reflecting the roots of the SU(4) group. The action of root reflections is encoded in the corresponding Weyl group, which for $SU(n)$ groups is S_n, the symmetric group on n elements. In the case of SU(4) we have S_4, which is of order 24 and is generated by three elementary reflections which act on the Dynkin labels as

$$\begin{aligned}
\sigma_1([d_1, d_2, d_3]) &= [-d_1, d_1 + d_2, d_3]\,, \\
\sigma_2([d_1, d_2, d_3]) &= [d_1 + d_2, -d_2, d_2 + d_3]\,, \\
\sigma_3([d_1, d_2, d_3]) &= [d_1, d_2 + d_3, -d_3]\,,
\end{aligned} \tag{3.7}$$

with the remaining elements given by various products of these, e.g. $\sigma_1\sigma_3$, $\sigma_1\sigma_2\sigma_1$, etc. What is relevant in our case is the so-called "shifted action" of the elements of the Weyl group, defined as

$$[d_1, d_2, d_3]^{\sigma_i} = \sigma_i([d_1 + 1, d_2 + 1, d_3 + 1]) - [1, 1, 1]\,, \tag{3.8}$$

yielding

$$\begin{aligned}
[d_1, d_2, d_3]^{\sigma_1} &= [-d_1 - 2, d_1 + d_2 + 1, d_3] \\
[d_1, d_2, d_3]^{\sigma_2} &= [d_1 + d_2 + 1, -d_2 - 2, d_2 + d_3 + 1] \\
[d_1, d_2, d_3]^{\sigma_3} &= [d_1, d_2 + d_3 + 1, -d_3 - 2],
\end{aligned} \tag{3.9}$$

In order to determine whether a negative-labeled representation contributes or not, we act on them with the "shifted action" (3.9) of all of the elements of the Weyl group. At most one of the elements, which can be expressed as a product of k elementary reflections, may render all the Dynkin labels positive. If such an element exists then this representation contributes to the tensor product with the transformed labels and an overall sign $(-1)^k$.

If no Weyl group element can turn all the labels positive then the representation does not contribute.[7] One can explicitly check that a representation will vanish if any of the following conditions on the initial labels holds:

$$\begin{array}{ccc}
d_1 = -1\,, & d_2 = -1\,, & d_3 = -1\,, \\
d_1 + d_2 = -2\,, & d_2 + d_3 = -2\,, & d_1 + d_2 + d_3 = -3\,.
\end{array} \tag{3.10}$$

We will now illustrate how this procedure works by an explicit calculation of the character for $\mathcal{V}_1^2$. The supermultiplet LWS is

$$\xi^A|0\rangle = \boxslash = \begin{pmatrix} {}^{1/2} & \mathbf{10} \\ \square, & 1 \end{pmatrix} + \begin{pmatrix} {}^{0} & \mathbf{15} \\ 1, & \square \end{pmatrix}\,, \tag{3.11}$$

[7]This "filtering" of negative-label representations to vanishing and contributing ones with the combined action $(-1)^k[d_1, d_2, d_3]^{\sigma_i}$ is what is more generally defined as the alternating Weyl sum, and it plays an essential part of the Racah-Speiser algorithm for decomposing tensor products into irreducible representations. See [45] for a general discussion, and the appendix of [46] for the particular application to SU(4). Here we have to consider this step for d_1 separately as the U(3)-invariant method of tensoring $S^{i\mu}$ with the LWS does not address it automatically.

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where we have also included the corresponding SU(2) Cartan charge j on top of the U(2) YT and the corresponding SU(4) multiplet on top of the U(3) YT.[8] Acting with one odd raising operator yields

$$\begin{aligned} S^{i\mu}\,\xi^A|0\rangle &= \left[\left(\ydiagram{2},\ydiagram{1,1}\right)+\left(\ydiagram{1,1},\ydiagram{2}\right)\right]\otimes\left[\left(\ydiagram{1},1\right)+\left(1,\ydiagram{1}\right)\right] \\ &= \left(\overset{1}{\ydiagram{2}},\overset{\mathbf{15}}{\ydiagram{1}}\right)+\left(\overset{0}{\ydiagram{1,1}},\overset{\mathbf{15}}{\ydiagram{1}}\right)+\left(\overset{1/2}{\ydiagram{1}},\overset{\overline{\mathbf{10}}}{\ydiagram{2}}\right)+\left(\overset{1/2}{\ydiagram{1}},\overset{\mathbf{6}}{\ydiagram{1,1}}\right). \end{aligned} \tag{3.12}$$

Continuing with a second odd raising operator we find

$$\begin{aligned} (S^{i\mu})^{\,2}\,\xi^A|0\rangle = &\left(\overset{3/2}{\ydiagram{3}},\overset{\mathbf{6}}{\ydiagram{1,1}}\right)+\left(\overset{1/2}{\ydiagram{2,1}},\overset{\mathbf{6}}{\ydiagram{1,1}}\right)+\left(\overset{1/2}{\ydiagram{2,1}},\overset{\overline{\mathbf{10}}}{\ydiagram{2}}\right)+\left(\overset{1}{\ydiagram{2}},\overset{\mathbf{1}}{\ydiagram{1,1,1}}\right) \\ &+\cancel{\left(\ydiagram{2},\ydiagram{2,1}\right)}+\cancel{\left(\ydiagram{1,1},\ydiagram{2,1}\right)}+\cancel{\left(\ydiagram{1,1},\ydiagram{3}\right)}, \end{aligned} \tag{3.13}$$

where the three last terms vanish because they all have $d_1 = f - l_1 - l_2 = -1$. From now on we will suppress such vanishing representations in all tensor products.

Moving on we find that the $Sp(4,\mathbb{R}) \times \mathrm{SO}(6)$ lowest-weight states arising from the action of more odd raising operators are

$$\begin{aligned} (S^{i\mu})^3\,\xi^A|0\rangle &= \left(\overset{2}{\ydiagram{4}},\overset{\mathbf{1}}{\ydiagram{1,1,1}}\right)+\left(\overset{1}{\ydiagram{3,1}},\overset{\mathbf{1}}{\ydiagram{1,1,1}}\right)+\left(\overset{1/2}{\ydiagram{2,1}},\overset{-\mathbf{6}}{\ydiagram{2,2}}\right)+\left(\overset{1/2}{\ydiagram{2,1}},\overset{-\overline{\mathbf{10}}}{\ydiagram{3,1}}\right), \\ (S^{i\mu})^4\,\xi^A|0\rangle &= \left(\overset{1}{\ydiagram{3,1}},\overset{-\mathbf{1}}{\ydiagram{2,2,1}}\right)+\left(\overset{1/2}{\ydiagram{3,2}},\overset{-\mathbf{6}}{\ydiagram{2,2}}\right)+\left(\overset{0}{\ydiagram{2,2}},\overset{-\mathbf{1}}{\ydiagram{2,2,1}}\right)+\left(\overset{0}{\ydiagram{2,2}},\overset{-\mathbf{15}}{\ydiagram{3,2}}\right), \\ (S^{i\mu})^5\,\xi^A|0\rangle &= \left(\overset{1}{\ydiagram{4,2}},\overset{-\mathbf{1}}{\ydiagram{2,2,1}}\right)+\left(\overset{0}{\ydiagram{3,3}},\overset{-\mathbf{1}}{\ydiagram{2,2,1}}\right), \\ (S^{i\mu})^6\,\xi^A|0\rangle &= \left(\overset{0}{\ydiagram{3,3}},\overset{\mathbf{1}}{\ydiagram{3,2,2}}\right), \end{aligned} \tag{3.14}$$

where in the SU(4) dimensionality we have also included the overall sign $(-1)^k$ coming from the action of the Weyl group elements. In more detail, we have for the particular negative-labeled U(3) Young tableaux

$$\begin{aligned} \ydiagram{2,2} &: [d_1,d_2,d_3] = [-2,2,0] \Rightarrow [d_1,d_2,d_3]^{\sigma_1} = [0,1,0], \quad (-1)^k = -1\,, \\ \ydiagram{3,1} &: [d_1,d_2,d_3] = [-2,1,2] \Rightarrow [d_1,d_2,d_3]^{\sigma_1} = [0,0,2], \quad (-1)^k = -1\,, \\ \ydiagram{2,2,1} &: [d_1,d_2,d_3] = [-2,1,0] \Rightarrow [d_1,d_2,d_3]^{\sigma_1} = [0,0,0], \quad (-1)^k = -1\,, \\ \ydiagram{3,2} &: [d_1,d_2,d_3] = [-3,2,1] \Rightarrow [d_1,d_2,d_3]^{\sigma_1} = [1,0,1], \quad (-1)^k = -1\,, \\ \ydiagram{3,2,2} &: [d_1,d_2,d_3] = [-3,0,1] \Rightarrow [d_1,d_2,d_3]^{\sigma_3\sigma_1} = [0,0,0], \quad (-1)^k = 1\,. \end{aligned} \tag{3.15}$$

[8] For compactness we indicate just the dimensionality of the representation as a proxy for its Dynkin labels: $\mathbf{10} = [2,0,0], \overline{\mathbf{10}} = [0,0,2], \mathbf{15} = [1,0,1], \mathbf{6} = [0,1,0], \mathbf{1} = [0,0,0]$.

Now summing all contributions (3.11)–(3.14) we find that the trace (3.4) (still restricted to the states in $\mathcal{V}_1^2$ that do not contain any derivatives P^{ij}) is

$$\begin{aligned}\tilde{\chi}_1^2 = {} & x^{3/2}\chi_{(1/2)}(\chi_{[200]} + \chi_{[002]}) \\ & + \big((x + x^2 - x^3)\chi_{(0)} + x^2\chi_{(1)}\big)\chi_{[101]} \\ & + \big((x^{3/2} - x^{7/2})\chi_{(1/2)} + x^{5/2}\chi_{(3/2)}\big)\chi_{[010]} \\ & + \big(- x^3\chi_{(0)} + (x^2 - x^4)\chi_{(1)} + x^3\chi_{(2)}\big)\chi_{[000]} \,,\end{aligned} \tag{3.16}$$

where

$$\chi(y)_{(j)} = \frac{y^{-j}(1 - y^{1+2j})}{1 - y} \tag{3.17}$$

is the character of the SU(2) $\subset Sp(4,\mathbb{R})$ representation with spin j and $\chi_{[d_1 d_2 d_3]}(u, r, v)$ is the character of the SU(4) representation with Dynkin labels $[d_1, d_2, d_3]$. The latter can be obtained by a slight modification of the U(4) character [36],

$$\chi(u, r, v)_{[d_1 d_2 d_3]} = \frac{\det(\epsilon_i^{n_l+4-l})}{\det(\epsilon_i^{4-l})} \tag{3.18}$$

where i and l label the rows and columns of a 4×4 matrix, the U(4) YT labels n_l are written in terms of the Dynkin labels according to

$$n_l = \sum_{i=l}^{3} d_i \qquad \text{for } l = 1, 2, 3\,, \qquad n_4 = 0 \tag{3.19}$$

for SU(4), and the variables ϵ_i are given by

$$\epsilon_1 = u\,, \qquad \epsilon_2 = \frac{r}{u}\,, \qquad \epsilon_3 = \frac{v}{r}\,, \qquad \epsilon_4 = \frac{1}{v} \tag{3.20}$$

so as to count units of the Cartan charges in the Dynkin basis. We notice that some powers of x in the character (3.16) have negative coefficients as a result of including the Weyl-transformed negative-label representations in (3.15). As explained in [46] these negative contributions are necessary to properly account for states which vanish due to equations of motion or due to conservation equations. A nice feature of including these states in the counting is that the full character, when states involving the spacetime derivatives P^{ij} are included, can be calculated naively with derivatives treated as if they acted freely. Since the derivatives have $\Delta = 1$ and belong to the spin $j = 1$ representation, summing over the set of states that arise by acting on an initial state with any number of derivatives introduces a multiplicative factor of $1/((1 - xy)(1 - x)(1 - x/y))$ into the character.

Finally then we arrive at the full character for $\mathcal{V}_1^2$,

$$\chi_1^2 = \frac{\tilde{\chi}_1^2}{(1 - x)(1 - xy)(1 - x/y)}\,, \tag{3.21}$$

with $\tilde{\chi}_1^2$ given in (3.16). In order to obtain the partition functions defined in (3.5) all we need to do is set u, r, v and y to 1, in which case the SU(2) and SU(4) characters reduce

1	$\mathcal{V}^2 = \mathcal{V}^2_{(1,0)[2,0,0]}$
$\begin{array}{c}\boxslash\\ \boxslash\end{array}$	$\mathcal{V}^2_{1,1} = \mathcal{V}^2_{(1,0)[0,0,2]} = \overline{\mathcal{V}}^2$
$\boxslash$	$\mathcal{V}^2_1 = \mathcal{V}^2_{(1,0)[1,0,1]}$
$\boxslash\boxslash$	$\mathcal{V}^2_2 = \mathcal{V}^2_{(1,0)[0,1,0]}$
$\overbrace{\boxslash\boxslash\cdots\boxslash}^{k\geq 3}$	$\mathcal{V}^2_k = \mathcal{V}^2_{(\frac{k-1}{2},\frac{k-3}{2})[0,0,0]}$

Table 1. The $f = 2$ $OSp(6|4)$ multiplets.

to the dimension formulas of the respective representations,

$$\begin{aligned}\chi(1)_{(j)} &= 2j+1\\ \chi(1,1,1)_{[d_1 d_2 d_3]} &= \frac{1}{12}(d_1+1)(d_2+1)(d_3+1)(d_1+d_2+2)(d_2+d_3+2)(d_1+d_2+d_3+3)\,.\end{aligned} \tag{3.22}$$

3.3 All irreducible multiplets with up to 4 sites

For a fixed value of f there exist only certain states that can be annihilated by all of the S_{AB} and hence serve as lowest-weight states for irreducible $OSp(6|4)$ representations. In this section we tabulate all possible representations with $f \leq 4$ together with their partition functions, obtained with the method described in the previous section. We have checked that all of the partition functions presented here are consistent with the general formulas presented in [26].

At $f = 1$ we have only the two fundamental representations of $OSp(6|4)$, called the 'singletons', which are conjugate to each other:

$$\left.\begin{aligned}&\mathcal{V}^1 = \mathcal{V}^1_{(\frac{1}{2},0)[1,0,0]}\\ &\mathcal{V}^1_{\boxslash} = \mathcal{V}^1_1 = \mathcal{V}^1_{(\frac{1}{2},0)[0,0,1]} = \overline{\mathcal{V}}^1\end{aligned}\right\} \quad V^1 = \overline{V}^1 = \frac{4\sqrt{x}}{(1-\sqrt{x})^2}\,. \tag{3.23}$$

As is well known, $\mathcal{V}^1$ includes the scalars of the ABJM theory and their supersymmetric partners, transforming respectively in the **4** and $\overline{\mathbf{4}}$ of SU(4). Note that obviously two conjugate representations will always have the same multiplicity of states at each energy level Δ, and hence will have equal partition functions.

Tables 1 and 2 display the possible multiplets for $f = 2$ and $f = 3$ respectively. The

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1	$\mathcal{V}^3 = \mathcal{V}^3_{(\frac{3}{2},0)[3,0,0]}$
$\begin{array}{c}\square\\\square\\\square\end{array}$	$\mathcal{V}^3_{1,1,1} = \mathcal{V}^3_{(\frac{3}{2},0)[0,0,3]} = \overline{\mathcal{V}}^3$
$\square$	$\mathcal{V}^3_1 = \mathcal{V}^3_{(\frac{3}{2},0)[2,0,1]}$
$\begin{array}{c}\square\\\square\end{array}$	$\mathcal{V}^3_{1,1} = \mathcal{V}^2_{(\frac{3}{2},0)[1,0,2]} = \overline{\mathcal{V}}^3_1$
$\square\square$	$\mathcal{V}^3_2 = \mathcal{V}^3_{(\frac{3}{2},0)[1,1,0]}$
$\begin{array}{l}\square\square\\\square\end{array}$	$\mathcal{V}^3_{2,1} = \mathcal{V}^3_{(\frac{3}{2},0)[0,1,1]} = \overline{\mathcal{V}}^3_2$
$\overbrace{\square\square\cdots\square}^{k\geq 3}$	$\mathcal{V}^3_k = \mathcal{V}^3_{(\frac{k}{2},\frac{k-3}{2})[1,0,0]}$
$\begin{array}{l}\overbrace{\square\square\cdots\square}^{k\geq 3}\\\square\end{array}$	$\mathcal{V}^3_{k,1} = \mathcal{V}^3_{(\frac{k}{2},\frac{k-3}{2})[0,0,1]} = \overline{\mathcal{V}}^3_k$

Table 2. The $f = 3$ $OSp(6|4)$ multiplets.

partition functions of these representations are given by

$$\begin{aligned} V^2 = V^2_{1,1} &= \frac{2x(5-x)}{(1-\sqrt{x})^3}\,, \\ V^2_1 &= \frac{x(15+7\sqrt{x}-3x-3x^{\frac{3}{2}})}{(1-\sqrt{x})^3}\,, \\ V^2_k &= x^{\frac{k-1}{2}}\frac{(1+\sqrt{x})^3}{(1-\sqrt{x})^3}\left(k-2+6\sqrt{x}-(k+2)x\right) \qquad k\geq 2\,, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} V^3 = V^3_{1,1,1} &= \frac{4x^{\frac{3}{2}}(5+3\sqrt{x})}{(1-\sqrt{x})^3}\,, \\ V^3_1 = V^3_{1,1} &= \frac{4x^{\frac{3}{2}}(9+11\sqrt{x}+4x)}{(1-\sqrt{x})^3}\,, \\ V^3_k = V^3_{k,1} &= \frac{4x^{\frac{k}{2}}(1+\sqrt{x})^3\,(k-2+(3+k)\sqrt{x})}{(1-\sqrt{x})^3} \qquad k\geq 2\,. \end{aligned} \tag{3.25}$$

The analysis for $f = 4$ is slightly more complicated since here the number of boxes in the second row of the SYT can be arbitrarily large. The possible representations are

1	$\mathcal{V}^4=\mathcal{V}_{(2,0)[4,0,0]}$		$\mathcal{V}^4_{1,1,1,1}=\mathcal{V}^4_{(2,0)[0,0,4]}=\overline{\mathcal{V}}^4$
	$\mathcal{V}^4_1=\mathcal{V}^4_{(2,0)[3,0,1]}$		$\mathcal{V}^4_{1,1,1}=\mathcal{V}^2_{(2,0)[1,0,3]}=\overline{\mathcal{V}}^4_1$
	$\mathcal{V}^4_2=\mathcal{V}^4_{(2,0)[2,1,0]}$		$\mathcal{V}^4_{2,1,1}=\mathcal{V}^4_{(2,0)[0,1,2]}=\overline{\mathcal{V}}^4_2$
$k\geq 3$	$\mathcal{V}^4_k=\mathcal{V}^4_{(\frac{k+1}{2},\frac{k-3}{2})[2,0,0]}$	$k\geq 3$	$\mathcal{V}^4_{k,1,1}=\mathcal{V}^4_{(\frac{k+1}{2},\frac{k-3}{2})[0,0,2]}=\overline{\mathcal{V}}^4_k$
	$\mathcal{V}^4_{1,1}=\mathcal{V}^4_{(2,0)[2,0,2]}$	$k\geq 3$	$\mathcal{V}^4_{k,1}=\mathcal{V}^4_{(\frac{k+1}{2},\frac{k-3}{2})[1,0,1]}$
	$\mathcal{V}^4_{2,1}=\mathcal{V}^4_{(2,0)[1,1,1]}$	$k\geq 3$	$\mathcal{V}^4_{k,2}=\mathcal{V}^4_{(\frac{k+1}{2},\frac{k-3}{2})[0,1,0]}$
	$\mathcal{V}^4_{2,2}=\mathcal{V}^4_{(2,0)[0,2,0]}$	$k_1\geq k_2$, $k_2\geq 3$	$\mathcal{V}^4_{k_1,k_2}=\mathcal{V}^4_{(\frac{k_1+k_2-2}{2},\frac{k_1-k_2}{2})[0,0,0]}$

Table 3. The $f=4$ $OSp(6|4)$ multiplets.

shown in table 3, while their partition functions are

$$\begin{aligned}
V^4=V^4_{1,1,1,1}&=\frac{x^2(35+35\sqrt{x}+9x+x^{\frac{3}{2}})}{(1-\sqrt{x})^3}\,,\\
V^4_1=V^4_{1,1,1}&=\frac{2x^2(35+59\sqrt{x}+36x+9x^{\frac{3}{2}}+x^2)}{(1-\sqrt{x})^3}\,,\\
V^4_{1,1}&=\frac{x^2(84+156\sqrt{x}+111x+39x^{\frac{3}{2}}+9x^2+x^{\frac{5}{2}})}{(1-\sqrt{x})^3}
\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}
V^4_k=V^4_{k,1,1}&=x^{\frac{k+1}{2}}\frac{(1+\sqrt{x})^3}{(1-\sqrt{x})^3}[10(k-2)+(k+1)(15\sqrt{x}+6x+x^{\frac{3}{2}})]\,,\\
V^4_{k,1}&=x^{\frac{k+1}{2}}\frac{(1+\sqrt{x})^3}{(1-\sqrt{x})^3}[15(k-2)+2(13k+6)\sqrt{x}+2(8k+1)x+6kx^{\frac{3}{2}}+kx^2]\,,\\
V^4_{k,2}&=x^{\frac{k+1}{2}}\frac{(1+\sqrt{x})^3}{(1-\sqrt{x})^3}[6(k-2)+4(4k-3)\sqrt{x}+(k-1)(20x+15x^{\frac{3}{2}}+6x^2+x^{\frac{5}{2}})]\,,\\
V^4_{k_1,k_2}&=(k_1-k_2+1)x^{\frac{k_1+k_2-2}{2}}\frac{(1+\sqrt{x})^9}{(1-\sqrt{x})^3}\,,
\end{aligned}\tag{3.27}$$

where $k\geq 2$ and $k_1\geq k_2\geq 3$.

4 Tensor product decompositions

In this section we compute tensor product decompositions for products of up to four copies of the $OSp(6|4)$ singletons $\mathcal{V}^1$, $\overline{\mathcal{V}}^1 = \mathcal{V}^1_1$. Before proceeding let us mention that although some of the results here are not immediately obvious, the correctness of all of the decompositions tabulated here can be verified with certainty using $OSp(6|4)$ characters and the relation $\chi_{\mathcal{V}_a\otimes\mathcal{V}_b} = \chi_{\mathcal{V}_a}\chi_{\mathcal{V}_b}$. We have performed this check using characters constructed according to the procedure outlined in section 3.2 or equivalently the expressions presented in [26].

4.1 Digraphs and syllables of the ABJM language

Continuing the linguistic analogy mentioned in the introduction we can think of the irreducible representations arising in the tensor product of two singletons as the 'digraphs' (see also [47]) of the ABJM language, groups of two successive letters whose phonetic value is a distinct sound, such as *aw* in *saw* or *qu* in *question*. We have found the consonant-vowel digraphs of the ABJM language to be

$$\mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 = \sum_{m=0}^{\infty} \mathcal{V}^2_{2m+1} = \mathcal{V}^2_{(1,0)[1,0,1]} + \sum_{n=0}^{\infty} \mathcal{V}^2_{(n+1,n)[0,0,0]}\,, \tag{4.1}$$

and we also mention the decomposition of a singleton squared, the consonant-consonant digraphs:

$$\mathcal{V}^1 \otimes \mathcal{V}^1 = \sum_{m=0}^{\infty} \mathcal{V}^2_{2m} = \mathcal{V}^2_{(1,0)[2,0,0]} + \mathcal{V}^2_{(1,0)[0,1,0]} + \sum_{n=0}^{\infty} \mathcal{V}^2_{(n+\frac{3}{2},n+\frac{1}{2})[0,0,0]}\,, \tag{4.2}$$

with $\overline{\mathcal{V}}^1 \otimes \overline{\mathcal{V}}^1$ simply being given by the conjugate of the latter.

Taking the analogy with linguistics further we can refer to the multiplets appearing in triple singleton products, on which the Hamiltonian density acts, as syllables, being the building blocks of words. Of most interest are the consonant-vowel-consonant syllables given by

$$\begin{aligned}\mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 \otimes \mathcal{V}^1 &= \sum_{m=0}^{\infty} (m+1)\left(\mathcal{V}^3_{2m+1} + \mathcal{V}^3_{2m+2,1}\right) \\ &= \mathcal{V}^3_{(\frac{3}{2},0)[2,0,1]} + \mathcal{V}^3_{(\frac{3}{2},0)[0,1,1]} + \sum_{n=0}^{\infty}(n+2)(\mathcal{V}^3_{(n+\frac{3}{2},n)[1,0,0]} + \mathcal{V}^3_{(n+2,n+\frac{1}{2})[0,0,1]})\end{aligned} \tag{4.3}$$

and we also mention the decomposition of a singleton cubed

$$\begin{aligned}\mathcal{V}^1 \otimes \mathcal{V}^1 \otimes \mathcal{V}^1 &= \sum_{m=0}^{\infty} (m+1)\left(\mathcal{V}^3_{2m} + \mathcal{V}^3_{2m+3,1}\right) \\ &= \mathcal{V}^3_{(\frac{3}{2},0)[3,0,0]} + 2\mathcal{V}^3_{(\frac{3}{2},0)[1,1,0]} \\ &\quad + \sum_{n=0}^{\infty}\left[(n+3)\mathcal{V}^3_{(n+2,n+\frac{1}{2})[1,0,0]} + (n+1)\mathcal{V}^3_{(n+\frac{3}{2},n)[0,0,1]}\right],\end{aligned} \tag{4.4}$$

again with the results for $\overline{\mathcal{V}}^1 \otimes \mathcal{V}^1 \otimes \overline{\mathcal{V}}^1$ and $\overline{\mathcal{V}}^1 \otimes \overline{\mathcal{V}}^1 \otimes \overline{\mathcal{V}}^1$ obviously obtained by conjugation.

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4.2 Four-fold tensor products

Although the three-fold tensor product decompositions presented above are sufficient for analyzing the two-loop Hamiltonian density, physical states of the ABJM spin chain must have an even number of sites to be gauge invariant so the shortest nontrivial 'words' of the ABJM language have length four.

The study of how these four-letter words arrange themselves into irreducible multiplets of the theory's $OSp(6|4)$ therefore deserves inquiry in its own right. This decomposition is more involved and we proceed by splitting the calculation into three steps. First we perform the decomposition of only three out of the four sites,

$$\begin{aligned} \mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 \otimes \mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 &= (\mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 \otimes \mathcal{V}^1) \otimes \overline{\mathcal{V}}^1 \\ &= \sum_{m=0}^{\infty} (m+1) \left(\mathcal{V}^3_{2m+1} \otimes \overline{\mathcal{V}}^1 + \mathcal{V}^3_{2m+2,1} \otimes \overline{\mathcal{V}}^1 \right), \end{aligned} \tag{4.5}$$

and then we perform the decomposition between the $f = 3$ irreducible multiplets $\mathcal{V}^3_k$ and $\mathcal{V}^3_{k,1}$ with $\overline{\mathcal{V}}^1$, for which we find

$$\begin{aligned} \mathcal{V}^3_{2n+1} \otimes \overline{\mathcal{V}}^1 &= \sum_{m=0}^{\infty} \sum_{j=0}^{n} \mathcal{V}^4_{2n+2m+2,2j} + \mathcal{V}^4_{2n+2m+1,2j+1}\,, \\ \mathcal{V}^3_{2n} \otimes \overline{\mathcal{V}}^1 &= \sum_{m=0}^{\infty} \left(\sum_{j=0}^{n} \mathcal{V}^4_{2n+2m+1,2j} + \sum_{j=0}^{n-1} \mathcal{V}^4_{2n+2m,2j+1} \right), \\ \mathcal{V}^3_{2n+1,1} \otimes \overline{\mathcal{V}}^1 &= \sum_{m=0}^{\infty} \left(\overline{\mathcal{V}}^4_{2n+2m+1} + \sum_{j=1}^{n} \mathcal{V}^4_{2n+2m+1,2j} + \sum_{j=1}^{n+1} \mathcal{V}^4_{2n+2m+2,2j-1} \right), \\ \mathcal{V}^3_{2n,1} \otimes \overline{\mathcal{V}}^1 &= \sum_{m=0}^{\infty} \left(\overline{\mathcal{V}}^4_{2n+2m} + \sum_{j=1}^{n} (\mathcal{V}^4_{2n+2m,2j} + \mathcal{V}^4_{2n+2m+1,2j-1}) \right). \end{aligned} \tag{4.6}$$

One can also combine the results for odd and even multiplet indices if desired,

$$\begin{aligned} \mathcal{V}^3_k \otimes \overline{\mathcal{V}}^1 &= \sum_{m=0}^{\infty} \left(\sum_{j=0}^{[\frac{k}{2}]} \mathcal{V}^4_{k+2m+1,2j} + \sum_{j=0}^{[\frac{k-1}{2}]} \mathcal{V}^4_{k+2m,2j+1} \right), \\ \mathcal{V}^3_{k,1} \otimes \overline{\mathcal{V}}^1 &= \sum_{m=0}^{\infty} \left(\overline{\mathcal{V}}^4_{2n+k} + \sum_{j=1}^{[\frac{k}{2}]} \mathcal{V}^4_{2n+k,2j} + \sum_{j=1}^{[\frac{k+1}{2}]} \mathcal{V}^4_{2n+k+1,2j-1} \right). \end{aligned} \tag{4.7}$$

Finally we have to extract the overall coefficient of each $V^4_{k_1,k_2}$ appearing in (4.5), which is facilitated by the fact that the coefficients of all the multiplets appearing in the direct sums (4.6) are all equal to one. A closer inspection reveals that $\mathcal{V}^4_{2j,2p}$ and $\mathcal{V}^4_{2j-1,2p-1}$ appear respectively in

- $\mathcal{V}^3_{2n+1} \otimes \overline{\mathcal{V}}^1$ for $p \le n \le j-1$ and $p-1 \le n \le j-1$, and

- $\mathcal{V}^3_{2n+2,1} \otimes \overline{\mathcal{V}}^1$ for $p-1 \leq n \leq j-1$ and $p-1 \leq n \leq j-2$.

It is clear from (4.5) that each $f = 4$ multiplet in question will receive a contribution of $(n+1)$ to its coefficient for each particular $\mathcal{V}^3_{2n+1} \otimes \overline{\mathcal{V}}^1$ or $\mathcal{V}^3_{2n+2,1} \otimes \overline{\mathcal{V}}^1$ in which it is contained, yielding in total the coefficients

$$\begin{aligned}
\mathcal{V}^4_{2j}, \overline{\mathcal{V}}^4_{2j}: &\qquad \sum_{n=0}^{j-1}(n+1) = \frac{1}{2}j(j+1)\,, \\
\mathcal{V}^4_{2j,2p}: &\qquad \sum_{n=p}^{j-1}(n+1) + \sum_{n=p-1}^{j-1}(n+1) = j^2+j-p^2\,, \\
\mathcal{V}^4_{2j-1,2p-1}: &\qquad \sum_{n=p-1}^{j-1}(n+1) + \sum_{n=p-1}^{j-2}(n+1) = j^2-p^2+p\,.
\end{aligned} \tag{4.8}$$

So putting everything together we find the following decomposition for the four-fold product of most interest,

$$\begin{aligned}
\mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 \otimes \mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 = &\sum_{j=1}^{\infty} \frac{1}{2}j(j+1)\left(\mathcal{V}^4_{2j} + \overline{\mathcal{V}}^4_{2j}\right) \\
&+ \sum_{j=1}^{\infty}\sum_{p=1}^{j}\left\{\left[j(j+1)-p^2\right]\mathcal{V}^4_{2j,2p} + \left[j^2-p(p-1)\right]\mathcal{V}^4_{2j-1,2p-1}\right\}.
\end{aligned} \tag{4.9}$$

For completeness we also mention the remaining four-fold product decompositions, which can be calculated in a similar fashion:

$$\begin{aligned}
\mathcal{V}^1 \otimes \mathcal{V}^1 \otimes \mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 = &\sum_{j=1}^{\infty} \frac{1}{2}j(j+1)\left(\mathcal{V}^4_{2j-1} + \overline{\mathcal{V}}^4_{2j+1}\right) \\
&+ \sum_{j=1}^{\infty}\sum_{p=1}^{j}\left\{\left[(j+1)^2-p^2\right]\mathcal{V}^4_{2j+1,2p} + \left[j(j+1)-p(p-1)\right]\mathcal{V}^4_{2j,2p-1}\right\},
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
\mathcal{V}^1 \otimes \mathcal{V}^1 \otimes \mathcal{V}^1 \otimes \mathcal{V}^1 = &\sum_{j=1}^{\infty} \frac{1}{2}j(j+1)\left(\mathcal{V}^4_{2j-2} + \overline{\mathcal{V}}^4_{2j+2}\right) \\
&+ \sum_{j=1}^{\infty}\sum_{p=1}^{j}\left\{\left[j(j+1)+1-p^2\right]\mathcal{V}^4_{2j,2p} + \left[j^2-1-p(p-1)\right]\mathcal{V}^4_{2j-1,2p-1}\right\}.
\end{aligned} \tag{4.11}$$

Notice that the very last term of the last relation actually has a vanishing coefficient for $j = p = 1$, but we have written it like this to coincide with the previous term's summation range. All other four-fold products may be obtained from (4.10), (4.11) or (4.12) by conjugation.

4.3 Four-letter words of the ABJM language

In the spin chain description of gauge theories only cyclically invariant spin chain states correspond to gauge-invariant, single-trace operators. For an ABJM spin chain with $f = 2L$ sites the physical states are those in the $+1$ eigenspace of the projection operator

$$\mathcal{P} = \frac{1}{L}(1 + T + T^2 + \cdots + T^{L-1}) = \frac{1}{L}\sum_{k=0}^{L-1} T^k\,, \tag{4.12}$$

expressed in terms of the translation operator T which sends site i to site $i+2$,

$$T|A_1B_1A_2\cdots A_LB_L\rangle = (-1)^{\deg(A_1B_1)\deg(A_2B_2\cdots A_LB_L)}|A_2B_2\cdots A_LB_LA_1B_1\rangle\,, \tag{4.13}$$

and obviously satisfies $T^L = 1$.

If we focus on the simplest nontrivial case $f = 2L = 4$, then $\mathcal{P} = \frac{1}{2}(1+T)$ and the physical subspace simply corresponds to the $+1$ eigenspace of T, which in turn consists of states $|w\rangle$ which are symmetric in the permutation of next-to-adjacent sites,

$$T(|A_1B_1A_2B_2\rangle + s|A_2B_2A_1B_1\rangle) = +1(|A_1B_1A_2B_2\rangle + s|A_2B_2A_1B_1\rangle)\,, \tag{4.14}$$

where $s = (-1)^{\deg(A_1B_1)\deg(A_2B_2)}$. Hence the set of physical states will be given by the symmetric square of $\mathcal{V}^1 \otimes \overline{\mathcal{V}}^1$, which can further be expressed as

$$(\mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 \otimes \mathcal{V}^1 \otimes \overline{\mathcal{V}}^1)_+ = (\mathcal{V}^1)^2_+ \otimes (\overline{\mathcal{V}}^1)^2_+ + (\mathcal{V}^1)^2_- \otimes (\overline{\mathcal{V}}^1)^2_-\,, \tag{4.15}$$

in terms of symmetric and antisymmetric 2-fold tensor products of $\mathcal{V}$ and $\overline{\mathcal{V}}$.

To proceed with the calculation of the symmetric and antisymmetric squares we use the general character formula (see for example [48])

$$\chi_{(\mathcal{V})^2_\pm}(g) = \frac{1}{2}[(\chi_{\mathcal{V}}(g))^2 \pm \chi_{\mathcal{V}}(g^2)]\,. \tag{4.16}$$

As pointed out in [36] this formula still holds for supergroups if we use supercharacters, defined by including $(-1)^F$ inside the trace,

$$\chi^S_{\mathcal{V}}(g) = \mathrm{Tr}_{\mathcal{V}}((-1)^F g)\,. \tag{4.17}$$

The oscillator construction makes clear a simple relation between $\chi^S_{\mathcal{V}}$ and $\chi_{\mathcal{V}}$ for any $\mathcal{V}$. This hinges on the observations that $(-1)^F$ is (perhaps confusingly) just $(-1)^{N_B}$, where N_B is the total boson number operator of (2.4), and that the bosonic and fermionic fields have Δ equal to $\frac{1}{2}$ and 1 respectively. Therefore from (2.6) we see that

$$(-1)^F x^\Delta = (-1)^{2\Delta - f} x^\Delta = (-1)^f (-\sqrt{x})^{2\Delta} \tag{4.18}$$

so that

$$\chi^S_{\mathcal{V}}(x^\Delta) = \chi^S_{\mathcal{V}}((\sqrt{x})^{2\Delta}) = (-1)^f \chi_{\mathcal{V}}((-\sqrt{x})^{2\Delta})\,. \tag{4.19}$$

This relation gives a general prescription for obtaining all supercharacters from the partition functions that we have already calculated in section 3.3.[9]

[9] Note that in (4.19) x denotes whatever is argument of the character that is exponentiated by Δ. When for example the argument is x^2, we would have to replace $x \to -x$.

Thus it is straightforward to apply the character formula (4.16) in order to calculate the symmetric and antisymmetric squares of $\mathcal{V}^1$, for which we find

$$(\mathcal{V}^1)^2_+ = \sum_{m=0}^{\infty} \mathcal{V}^2_{4m}\,, \qquad (\mathcal{V}^1)^2_- = \sum_{m=0}^{\infty} \mathcal{V}^2_{4m+2}\,, \tag{4.20}$$

with the respective relations for $\overline{\mathcal{V}}^1$ obtained from these by conjugation (note that $\overline{\mathcal{V}}^2_{2j} = \mathcal{V}^2_{2j}$ unless $j = 0$). Clearly the next step involves decomposing tensor products of the form $\mathcal{V}^2_{2l} \otimes \overline{\mathcal{V}}^2_{2m}$, for which we find

$$\begin{aligned}
\mathcal{V}^2_0 \otimes \overline{\mathcal{V}}^2_0 &= \sum_{p=0}^{\infty}\sum_{j=p}^{\infty} \mathcal{V}^4_{2j+1,2p+1}\,,\\
\mathcal{V}^2_0 \otimes \overline{\mathcal{V}}^2_{2m} &= \sum_{p=0}^{\infty}\sum_{j=p+m}^{\infty} \mathcal{V}^4_{2j+1,2p+1} + \mathcal{V}^4_{2j,2p}\,,\\
\mathcal{V}^2_{2l} \otimes \overline{\mathcal{V}}^2_{2m} &= \sum_{j=l+m}^{\infty} (\mathcal{V}^4_{2j} + \overline{\mathcal{V}}^4_{2j})\,,\\
&\quad + \sum_{p=1}^{m}\sum_{j=l+m-p}^{\infty} c(j-l-m-p)\mathcal{V}^4_{2j,2p} + c(j+1-l-m-p)\mathcal{V}^4_{2j+1,2p-1}\\
&\quad + \sum_{p=m+1}^{\infty}\sum_{j=l+p-m}^{\infty} c(j-l-m-p)(\mathcal{V}^4_{2j,2p} + \mathcal{V}^4_{2j-1,2p-1})\,,
\end{aligned} \tag{4.21}$$

where $l \geq m \geq 1$ and

$$c(k) = 1 + \Theta(k) = \begin{cases} 1 & \text{if } k < 0\,,\\ 2 & \text{if } k \geq 0\,,\end{cases} \tag{4.22}$$

and we use $\Theta(k)$ to denote the unit step function.

Combining all of these intermediate steps we deduce that the set of physical states for the $f = 2L = 4$ spin chain decomposes into irreducible $OSp(6|4)$ multiplets as follows:

$$\begin{aligned}
(\mathcal{V}^1 \otimes \overline{\mathcal{V}}^1)^2_+ &= \sum_{j=1}^{\infty} j(j+1)(\mathcal{V}^4_{4j} + \overline{\mathcal{V}}^4_{4j} + \mathcal{V}^4_{4j+2} + \overline{\mathcal{V}}^4_{4j+2}) + [2j(j-1)+1]\mathcal{V}^4_{4j-3,1}\\
&\quad + \sum_{j=1}^{\infty}\sum_{p=1}^{j} \Big\{2\left[j(j+1)-p^2\right](\mathcal{V}^4_{4j,4p} + \mathcal{V}^4_{4j+2,4p})\\
&\qquad + 2\left[j^2 - p(p-1)\right](\mathcal{V}^4_{4j-1,4p-1} + \mathcal{V}^4_{4j-1,4p-3})\\
&\qquad + \left[2j^2 - 1 - 2p(p-1)\right](\mathcal{V}^4_{4j-2,4p-2} + \mathcal{V}^4_{4j,4p-2})\\
&\qquad + \left[2j(j+1) + 1 - 2p^2\right](\mathcal{V}^4_{4j+1,4p-1} + \mathcal{V}^4_{4j+1,4p+1})\Big\}\,.
\end{aligned} \tag{4.23}$$

5 The $OSp(4|2)$ subsector

In this section we tabulate for completeness various results from the previous two sections for the case of the $OSp(4|2) \subset OSp(6|4)$ subgroup. This is of particular interest since the two-loop dilatation operator has been constructed explicitly in terms of its action on the fundamental fields in this subsector of the ABJM theory by Zweibel [22].

The oscillator construction we reviewed in section 2 may be restricted to the $OSp(4|2)$ subsector simply by restricting the bosonic and fermionic oscillator indices to $i = 1$ and $\mu = 1, 2$ respectively, which gives the bosonic subgroup $Sp(2, \mathbb{R}) \times \mathrm{SO}(4)$. The $Sp(2, \mathbb{R}) \simeq \mathrm{SU}(2) \simeq \mathrm{SO}(3)$ content of a LWS is characterized just by its scaling dimension $\Delta = \frac{1}{2}(N_{B_1} + f)$, and the $\mathrm{SO}(4) \simeq \mathrm{SO}(3) \times \mathrm{SO}(3)$ content is characterized by the Dynkin labels $[p, q]$ which are simply the Cartan charges of each of the two SO(3)s. In each $Sp(2, \mathbb{R})$ multiplet there exists now only a single state for each value of Δ, whereas the dimensionality of an SO(4) multiplet $[p, q]$ is $(p+1)(q+1)$. Similarly to (3.3), the $OSp(4|2)$ multiplet with SYT labels $(k_1, \dots, k_n)$ will have Cartan labels

$$\begin{aligned}
\Delta &= \frac{1}{2}\left(\max(k_1 - 2, 0) + f\right) , \\
p &= f - \sum_{i=1}^{n} \min(k_i, 2) , \\
q &= \sum_{i=1}^{n} \delta_{k_i,1} .
\end{aligned} \tag{5.1}$$

5.1 Partition functions

The SYT labeling of supermultiplets is convenient here as well, and it turns out that for each value of $f \leq 3$ we get the precisely the same type of multiplets as in the $OSp(6|4)$ cases considered above. The partition functions for all of the $f \leq 3$ multiplets are

$$\begin{aligned}
V^1 = V^1_1 &= \frac{2\sqrt{x}}{1-\sqrt{x}} , \\
V^2 = V^2_{1,1} &= \frac{x(3+\sqrt{x})}{1-\sqrt{x}} , & V^3 = V^3_{1,1,1} &= \frac{2x^{\frac{3}{2}}(2+\sqrt{x})}{1-\sqrt{x}} , \\
V^2_1 &= \frac{x(4+3\sqrt{x}+x)}{1-\sqrt{x}} , & V^3_1 = V^3_{1,1} &= \frac{2x^{\frac{3}{2}}(3+3\sqrt{x}+x)}{1-\sqrt{x}} , \\
V^2_k &= \frac{x^{\frac{k}{2}}(1+\sqrt{x})^3}{1-\sqrt{x}} , & V^3_k = V^3_{k,1} &= \frac{2x^{\frac{k+1}{2}}(1+\sqrt{x})^3}{1-\sqrt{x}} \qquad k \geq 2 .
\end{aligned} \tag{5.2}$$

When we move to $f = 4$ however, the smaller number of superoscillator components compared to $OSp(6|4)$ reduces the number of possible supercovariant symmetrizations and antisymmetrizations, leaving us with only a subset of the types of multiplets which appeared above. Specifically, only those multiplets whose lowest-weight states have $k_2 \leq 2$ boxes in the second row of their super-Young tableaux are now allowed. The partition functions of these multiplets are

$$
\begin{aligned}
V^4 = V^4_{1,1,1,1} &= \frac{x^2(5+3\sqrt{x})}{1-\sqrt{x}}\,, \\
V^4_1 = V^4_{1,1,1} &= \frac{x^2(8+9\sqrt{x}+3x)}{1-\sqrt{x}}\,, \\
V^4_{1,1} &= \frac{x^2(9+11\sqrt{x}+4x)}{1-\sqrt{x}}\,, \\
V^4_k = V^4_{k,1,1} &= \frac{3x^{\frac{k+2}{2}}(1+\sqrt{x})^3}{1-\sqrt{x}}\,, \\
V^4_{k,1} &= \frac{4x^{\frac{k+2}{2}}(1+\sqrt{x})^3}{1-\sqrt{x}}\,, \\
V^4_{k,2} &= \frac{x^{\frac{k+2}{2}}(1+\sqrt{x})^3}{1-\sqrt{x}}\,,
\end{aligned}
\tag{5.3}
$$

where $k \geq 2$. We notice that the last three partition functions are actually proportional to each other, however the SO(4) content of the corresponding representations is not the same.

5.2 Tensor products

As far as the tensor products are concerned, due to the one-to-one correspondence of multiplets for $f \leq 3$ we find identical results (when the multiplets are expressed in SYT notation) to those presented in (4.1) through (4.4). For $f = 4$, the existence of fewer multiplets simplifies the decompositions slightly to

$$
\begin{aligned}
\mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 \otimes \mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 &= \sum_{j=1}^{\infty} \frac{1}{2} j(j+1) \left(\mathcal{V}^4_{2j} + \overline{\mathcal{V}}^4_{2j}\right) + \left(j^2+j-1\right) \mathcal{V}^4_{2j,2} + j^2 \mathcal{V}^4_{2j-1,1}\,, \\
\mathcal{V}^1 \otimes \mathcal{V}^1 \otimes \mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 &= \sum_{j=1}^{\infty} \frac{1}{2} j(j+1) \left(\mathcal{V}^4_{2j-1} + \overline{\mathcal{V}}^4_{2j+1}\right) + j(j+2) \mathcal{V}^4_{2j+1,2} + j(j+1) \mathcal{V}^4_{2j,1}\,, \\
\mathcal{V}^1 \otimes \mathcal{V}^1 \otimes \mathcal{V}^1 \otimes \mathcal{V}^1 &= \sum_{j=1}^{\infty} \frac{1}{2} j(j+1) \left(\mathcal{V}^4_{2j-2} + \overline{\mathcal{V}}^4_{2j+2}\right) + j(j+1) \mathcal{V}^4_{2j,2} + j(j+2) \mathcal{V}^4_{2j+1,1}\,.
\end{aligned}
\tag{5.4}
$$

It is evident that the $OSp(4|2)$ four-fold relations above can be obtained from the respective $OSp(6|4)$ ones (4.10) through (4.12) by restricting the summation on p to $p \leq 1$ or equivalently to $k_2 \leq 2$, which is reasonable as $OSp(4|2)$ does not contain any representations with $k_2 > 2$. Another way to see it is that if we tried to take the linear combinations of superoscillators corresponding to an $OSp(6|4)$ LWS with $k_2 > 2$ by using just the subset of $OSp(4|2)$ oscillators, we would get a vanishing result.

Finally the symmetrized self-conjugate 4-fold product analogous to the $OSp(6|4)$ result (4.23) is now

$$
\begin{aligned}
\left(\mathcal{V}^1 \otimes \overline{\mathcal{V}}^1\right)^2_+ = \sum_{j=1}^{\infty} &\lfloor \tfrac{j+1}{2} \rfloor \left(\lfloor \tfrac{j+1}{2} \rfloor + 1\right) \left(\mathcal{V}^4_{2j+2} + \overline{\mathcal{V}}^4_{2j+2}\right) \\
&+ \left(2\lfloor \tfrac{j+1}{2} \rfloor^2 - 1\right) \mathcal{V}^4_{2j,2} + \lfloor \tfrac{j^2+1}{2} \rfloor \mathcal{V}^4_{2j-1,1}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty} j(j+1) \left(\mathcal{V}^4_{4j} + \overline{\mathcal{V}}^4_{4j} + \mathcal{V}^4_{4j+2} + \overline{\mathcal{V}}^4_{4j+2} \right) \\
&\quad +(2j^2-1) \left(\mathcal{V}^4_{4j-2,2} + \mathcal{V}^4_{4j,2} \right) \\
&\quad +(2j(j-1)+1)\mathcal{V}^4_{4j-3,1} + 2j^2 \mathcal{V}^4_{4j-1,1} \, ,
\end{aligned} \tag{5.5}
$$

where $\lfloor m \rfloor$ denotes the integer part of m. Again the $OSp(4|2)$ result (5.5) follows from the $OSp(6|4)$ one (4.23) by simply restricting to $k_2 \le 2$.

6 A first peek at the two-loop dilatation operator

Much of this paper has been rather encyclopedic in nature so before concluding we present here an example of a result which follows relatively easily from information tabulated in the preceding sections. Specifically, we use the explicit form of the two-loop dilatation operator [22, 23] to calculate a certain trace $\langle D_2(x) \rangle$ of the Hamiltonian density. This quantity enters into the formula [47] for the (in this case) two-loop correction to the partition function of planar ABJM theory on S^2. It is known [49] that, like planar SYM on S^3 [50, 51], the theory has a Hagedorn temperature $T_{\rm H}$, which is a non-trivial function of the 't Hooft coupling λ, and the two-loop correction to $T_{\rm H}$ at weak coupling is determined by $\langle D_2(-1/\log T_{\rm H}) \rangle$. See [34, 52–54] for other work on partition functions for M2-brane theories.

6.1 The trace $\langle D_2(x) \rangle$ of the Hamiltonian density

The two-loop dilatation operator acting on a spin chain state of length $2L$ in the ABJM theory takes the form [22, 23]

$$
\Delta_2 = \lambda^2 \sum_{i=1}^{2L} (D_2)_{i,i+1,i+2} \tag{6.1}
$$

where the Hamiltonian density D_2 acts simultaneously on three adjacent sites of the chain according to

$$
\begin{aligned}
(D_2)_{123} = &\sum_{j=0}^{\infty} h(j) \mathcal{P}^{(j)}_{12} \\
&+ \sum_{j_1,j_2,j_3=0}^{\infty} (-1)^{j_1+j_3} \left(\tfrac{1}{2} h(j_2 - \tfrac{1}{2}) + \log 2 \right) \\
&\times \left(\mathcal{P}^{(j_1)}_{12} \mathcal{P}^{(j_2-1/2)}_{13} \mathcal{P}^{(j_3)}_{12} + \mathcal{P}^{(j_1)}_{23} \mathcal{P}^{(j_2-1/2)}_{13} \mathcal{P}^{(j_3)}_{23} \right) ,
\end{aligned} \tag{6.2}
$$

where $h(j)$ are harmonic numbers and $\mathcal{P}^{(j)}_{ab}$ is the projection operator whose image is spanned by states with $OSp(6|4)$ spin j (see [22] for details) in the tensor product space of sites a and b.

The trace we are interested in computing is

$$
\langle D_2(x) \rangle = \mathrm{Tr}_{\mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 \otimes \mathcal{V}^1} [x^{\Delta} D_2] \, . \tag{6.3}
$$

As a consequence of $OSp(6|4)$ symmetry we can make use of the tensor product decomposition (4.4), rewritten slightly here as

$$\mathcal{V}^1 \otimes \overline{\mathcal{V}}^1 \otimes \mathcal{V}^1 = \sum_{n=0}^{\infty} \lfloor \tfrac{n+1}{2} \rfloor \mathcal{V}(n) \,, \qquad \mathcal{V}(n) \equiv \begin{cases} \mathcal{V}^3_n & \text{for } n \text{ odd} \,, \\ \mathcal{V}^3_{n,1} & \text{for } n \text{ even} \,. \end{cases} \tag{6.4}$$

to conclude that D_2 can be brought to the block-diagonal form

$$D_2 = \sum_{n=0}^{\infty} M_n \otimes \mathcal{P}_n \tag{6.5}$$

where $\mathcal{P}_n$ is the projection operator whose image consists of the union of the $\lfloor \frac{n+1}{2} \rfloor$ copies of $\mathcal{V}(n)$ appearing in (6.4) and M_n is an $\lfloor \frac{n+1}{2} \rfloor \times \lfloor \frac{n+1}{2} \rfloor$ matrix. This form makes it clear that the desired trace can be calculated as

$$\langle D_2(x) \rangle = \sum_{n=0}^{\infty} \mathrm{Tr}[M_n] \, V(n) \tag{6.6}$$

in terms of the characters which may be read off from (3.25)—note in particular that $V(n) = V^3_n = V^3_{n,1}$ for $n \geq 2$. From (6.2) we obtain the values

$$\mathrm{Tr}[M_n] = \begin{cases} 2 \displaystyle\sum_{j=0}^{(n-2)/2} h(2j+1), & n \text{ even} \,, \\ 2 \displaystyle\sum_{j=0}^{(n-1)/2} h(2j) & n \text{ odd} \,, \end{cases} \tag{6.7}$$

which in turn lead to the result

$$\langle D_2(x) \rangle = 8\sqrt{x} \frac{(1+\sqrt{x})^2}{(1-\sqrt{x})^6} \left[\sqrt{x} + x + (1 - 6\sqrt{x} + x) \log(1-\sqrt{x}) \right] . \tag{6.8}$$

6.2 The two-loop hagedorn temperature

At zero 't Hooft coupling and infinite N the partition function $\mathcal{Z}$ of the ABJM theory with gauge group $\mathrm{U}(N) \times \mathrm{U}(N)$ on an S^2 of radius 1 can be expressed (see for example [34, 49, 52, 54]) as

$$\log \mathcal{Z}(x) \Big|_{\lambda=0} = \log \mathrm{Tr}[x^\Delta] \Big|_{\lambda=0} = -\sum_{n=1}^{\infty} \log[1 - z(\omega^{n+1} x^n)^2] \,, \qquad x = e^{-1/T} \,, \tag{6.9}$$

where the trace is taken over the full Hilbert space of the ABJM theory, $z(x) = \frac{4\sqrt{x}}{(1-\sqrt{x})^2}$ is the singleton partition function and $\omega = e^{2\pi i}$ is a convenient bookkeeping device which is defined to take the value $\sqrt{\omega^m} = +1$ (-1) if m is even (odd). The expression (6.2) is valid in the low-temperature phase $x < x_{\mathrm{H}}$, where the Hagedorn value $x_{\mathrm{H}} = 17 - 12\sqrt{2}$ is the smaller solution of $z(x) = 1$.

The two-loop correction follows from the general analysis of [47] (see also [55, 56] for other applications) and takes the form[10]

$$\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\log \mathrm{Tr}[x^\Delta]\Big|_{\lambda=0} \sim -\log x \sum_{n=1}^{\infty} n \frac{\langle D_2(\omega^{n+1}x^n)\rangle z(\omega^{n+1}x^n)}{1-z(\omega^{n+1}x^n)^2} \tag{6.10}$$

where $\sim$ denotes that we have omitted some additional terms which are negligibly small as we approach the pole in the partition function $x \to x_{\mathrm{H}}$ from below. It follows from (6.10) that the $\mathcal{O}(\lambda^2)$ correction δT_{H} to the Hagedorn temperature is

$$\frac{\delta T_{\mathrm{H}}}{T_{\mathrm{H}}} = \frac{\lambda^2}{\sqrt{2}}\langle D_2(x_{\mathrm{H}})\rangle = 2\lambda^2(\sqrt{2}-1)\,, \tag{6.11}$$

using (6.8).

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A Decomposition of $OSp(6|4)$ super-Young tableaux

As we mentioned in section 2.3, the LWS of each $OSp(6|4)$ multiplet belongs to a certain $U(2|3)$ representation, which can be conveniently labeled by its SYT. In turn, each $U(2|3)$ representation can be decomposed into a set of $\mathrm{U}(2)\times\mathrm{U}(3)$ representations labeled by their respective ordinary Young tableaux.

In terms of superoscillators, this decomposition (called branching) simply amounts to restricting their superindices to taking either only bosonic or fermionic values in all possible ways with distinct symmetrization and antisymmetrization properties. This way one can immediately perform the decompositions of the first few cases,

$$\begin{aligned}
\boxslash &= (\square, 1) + (1, \square)\,,\\
\begin{smallmatrix}\boxslash\\\boxslash\end{smallmatrix} &= \left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, 1\right) + (\square, \square) + (1, \square\square)\,,\\
\boxslash\boxslash &= (\square\square, 1) + (\square, \square) + \left(1, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\,,\\
\overbrace{\boxslash\boxslash\cdots\boxslash}^{k\geq 3} &= \left(\overbrace{\square\square\cdots\square}^{k}, 1\right) + \left(\overbrace{\square\square\cdots\square}^{k-1}, \square\right) + \left(\overbrace{\square\square\cdots\square}^{k-2}, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right) + \left(\overbrace{\square\square\cdots\square}^{k-3}, \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right)\,,
\end{aligned} \tag{A.1}$$

[10] If the deconfinement transition of the ABJM theory at small λ is first-order, as for example is the case [57] for Yang-Mills theory on a small S^3, then $\mathcal{Z}(x)$ will begin to diverge from $\mathrm{Tr}[x^\Delta]$ for x slightly below x_{H}, but this does not affect our calculation of the Hagedorn temperature.

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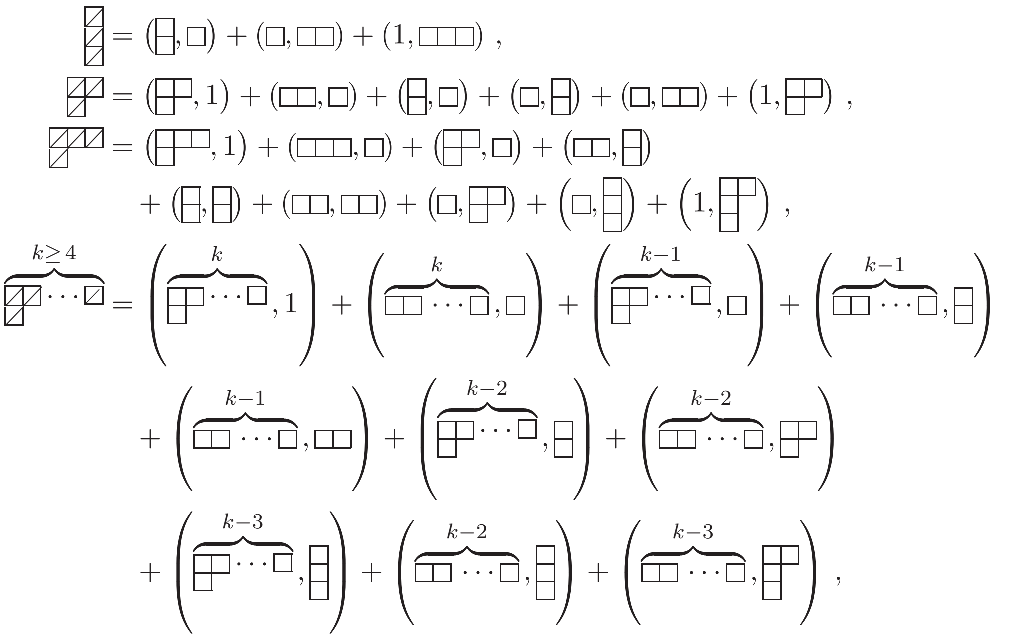
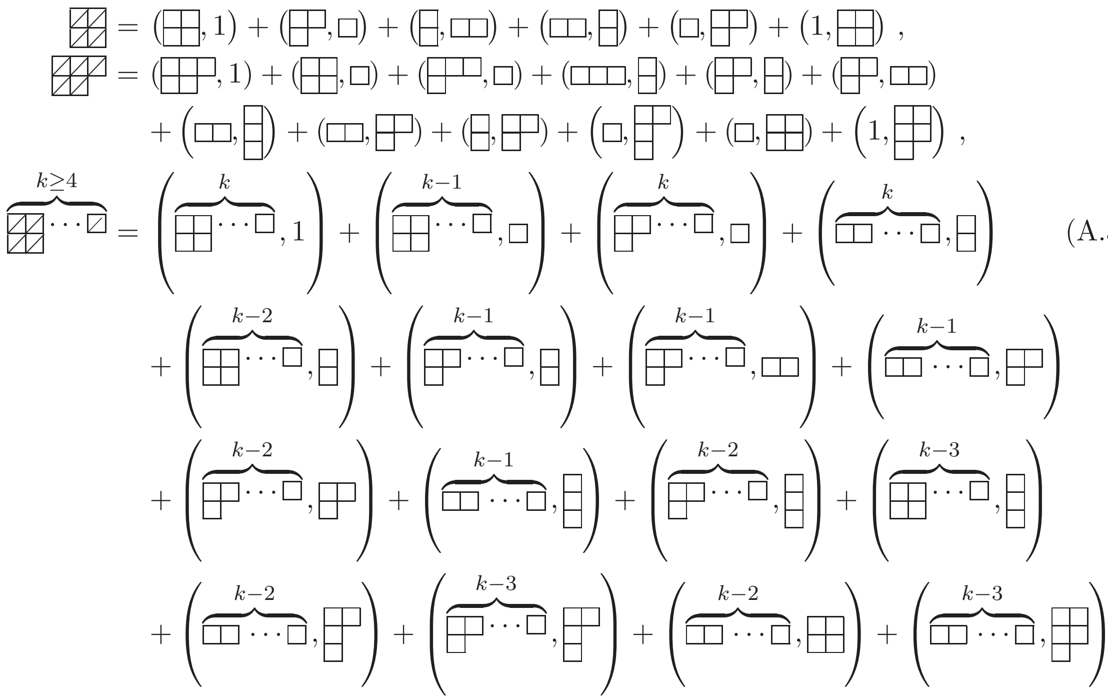
which account for all the possible lowest-weight states which can appear for $f = 1$ and $f = 2$, as one can see in section 3.3. For more complicated cases one needs to use the general formulas arising from the one-to-one correspondence between the $\mathrm{SU}(N+M)$ and $SU(N|M)$ decompositions to $\mathrm{SU}(N) \times SU(M) \times \mathrm{U}(1)$ established in [38], or alternatively use the set of SYT decomposition rules mentioned in [35].

Here we provide a detailed list of the the additional SYT that appear for $f = 3, 4$ lowest-weight states, together with their $\mathrm{U}(2) \times \mathrm{U}(3)$ decompositions. In particular, for $f = 3$ we can have in addition to (A.1) the super-Young tableaux



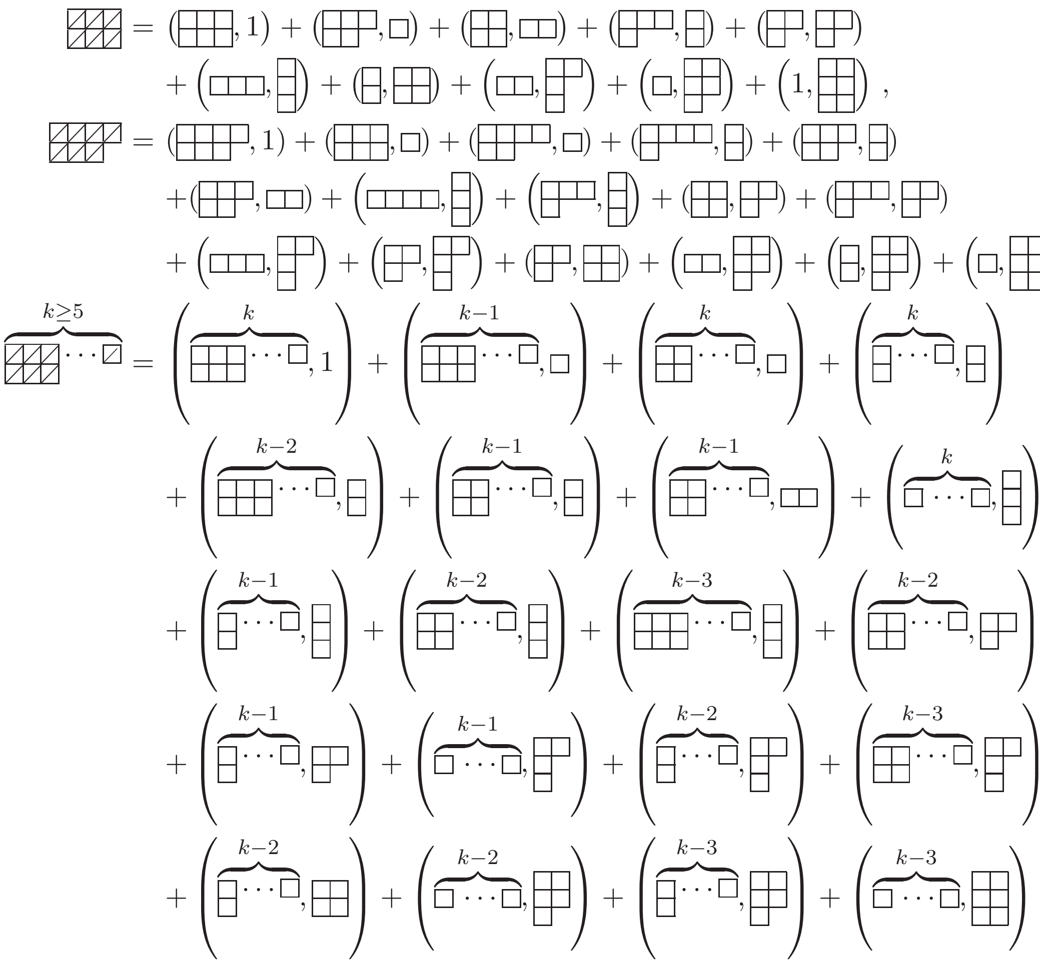
(A.2)

while for $f = 4$ we can also have



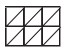
(A.3)

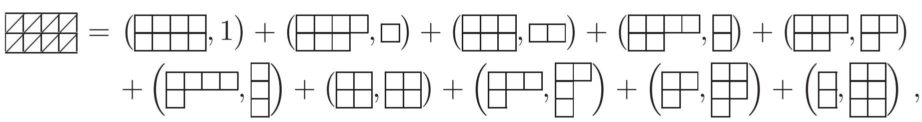
and



.(A.4)

In the last lines of (A.3) and (A.4) it should be understood that a representation should be omitted from the right-hand side if the number of boxes in the first row is less than the number in the second. Also, we have omitted the SYT of multiplets which can be obtained from the ones shown by conjugation, such as  and so on.

Finally, it can be shown that $U(2|3)$ representations with $k_2 = 3{+}m$ boxes in the second row of their SYT have identical decompositions as representations with $k_2 = 3$ except that they have m additional two-box columns in their respective U(2) Young tableaux, or alternatively in terms of quantum numbers only the scaling dimension changes as $\Delta \to \Delta + m$. For example, compare the decomposition of  above with



(A.5)

which evidently differs only by the addition of a single two-box column to the U(2) part of the decomposition. With this rule one can easily obtain the decompositions of the remaining $f = 4$ SYT with $k_2 > 3$ from the results given in (A.4).

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